

Self force on a scalar charge in the spacetime of a stationary, axisymmetric black hole

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We study the self force acting on a particle endowed with scalar charge, which is held static (with respect to an undragged, static observer at infinity) outside a stationary, axially-symmetric black hole. We find that the acceleration due to the self force is in the same direction as the black hole's spin, and diverges when the particle approaches the outer boundary of the black hole's ergosphere. This acceleration diverges more rapidly approaching the ergosphere's boundary than the particle's acceleration in the absence of the self force. At the leading order this self force is a (post)²-Newtonian effect. For scalar charges with high charge-to-mass ratio, the acceleration due to the self force starts dominating over the regular acceleration already far from the black hole. The self force is proportional to the rate at which the black hole's rotational energy is dissipated. This self force is local (i.e., only the Abraham-Lorentz-Dirac force and the local coupling to Ricci curvature contribute to it). The non-local, tail part of the self force is zero.

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I. INTRODUCTION AND SUMMARY

Recently there has been much interest in the calculation of self interaction of particles in curved spacetime. This growing interest is motivated by the prospects of detection of low-frequency gravitational waves in the not-so-distant future by space-borne gravitational wave detectors such as LISA. The main challenge is to compute the orbital evolution, and the resulting wave forms, from a compact object orbiting a supermassive central black hole in the extreme mass ratio case. The motion of the compact object, which may be construed as a structureless particle, is geodesic in the limit of zero mass. However, when the particle is endowed with a finite (albeit small) mass, its motion is changed. Spacetime is now determined by the energy-momenta of both the black hole and the particle, and the latter follows then a geodesic of the new spacetime, which is perturbed by its own energy-momentum relative to the original spacetime. An alternative viewpoint is to consider the motion of the particle as an accelerated, non-geodesic motion in the unperturbed spacetime of the central black hole. Whereas this latter approach is less in the spirit of Einstein's General Relativity, which "eliminated" gravitational forces in favor of geometry, than the former approach, it has the advantage that the unperturbed spacetime is often very simple. (For many interesting cases it is, e.g., the stationary and axially-symmetric spacetime of a Kerr black hole.) We are thus led naturally to translate the problem of finding the orbital evolution of the particle to the following question: what are all the momentary forces which act on the compact object? (In the absence of external forces, there would be just the self force.) Obviously, knowledge of all the forces which act on an object in a given spacetime allows for the computation of the orbit and consequently also for the computation of the emitted gravitational waves.

The problem of finding the self force which acts on a particle in curved spacetime is not easy. The reason is that in curved spacetime, due to the failure of the Huygens principle, the retarded Green's function associated with the particle's field has support inside the future light cone, and in particular also on the future world line. (The physical origin for this phenomenon is the scattering of the emitted waves off the spacetime curvature.) The part of the Green's function inside the light cone has been dubbed its "tail" part, and it is the calculation of this tail part of the retarded Green's function which is the greatest challenge in the computation of the self force (a.k.a. the radiation reaction force). A difficulty associated with the calculation of the tail part of the Green's function is the prescription used in order to separate the tail part, which affects the motion of the particle, from the instantaneous, divergent part, which arises from the typical divergence of the particle's field in the coincidence limit of the source for the field and the evaluation point.

Several approaches have been proposed for the regularization of the self force (SF). One approach was first suggested by Dirac for a pointlike electric charge in flat spacetime [1], and later used by DeWitt and Brehme for the case of an electric charge in curved spacetime [2] and by Mino, Sasaki, and Tanaka for the case of a pointlike particle coupled to linearized gravity [3]. The idea is to impose local energy and momentum conservation on a tube surrounding the particle's world line, and to integrate the conservation laws across the tube, thus obtaining the particle's equations of motion, including the SF effects. The divergent piece of the self force is then removed by a mass-regularization procedure. A second, axiomatic approach, which leads to the same expression as the approach described above, was proposed by Quinn and Wald for electromagnetic and gravitational self forces [4] and by Quinn for the scalar field case

[5]. According to this approach, the regularization of the SF is performed by comparing the forces in two different spacetimes.

For the case of a particle coupled to a (minimally-coupled, massless) scalar field, the total SF which acts on the particle is given by [5]

$$F_{\mu}^{\text{SF}} = \frac{1}{3}q^2(\ddot{u}_{\mu} - u_{\mu}\dot{u}_{\alpha}\dot{u}^{\alpha}) + \frac{1}{6}q^2\mathcal{R}_{\mu} + \lim_{\epsilon \rightarrow 0^-} q^2 \int_{-\infty}^{\tau+\epsilon} d\tau' \nabla_{\mu} G^{\text{ret}}[z^{\alpha}(\tau), z^{\alpha}(\tau')], \quad (1)$$

where $\mathcal{R}_{\mu} = R_{\mu\alpha}u^{\alpha} + R_{\alpha\beta}u^{\alpha}u^{\beta}u_{\mu} - \frac{1}{2}Ru_{\mu}$. Here, $R_{\mu\nu}$ is the Ricci tensor, R is the curvature scalar, and u^{α} is the particle's four-velocity. An overdot denotes (covariant) differentiation with respect to proper time τ , q is the particle's scalar charge, $G^{\text{ret}}[z^{\alpha}(\tau), z^{\alpha}(\tau')]$ is the retarded Green's function, and $z^{\alpha}(\tau)$ is the particle's world line. The total force which acts on the particle is the sum of external forces (e.g., forces which result from external scalar fields) and the SF (1). The SF in Eq. (1) has three contributions: the first is a local, Abraham-Lorentz-Dirac (ALD) type force. The ALD force consists of two terms, a term proportional to the proper time derivative of the four-acceleration, called the ‘‘Schott part’’ of the ALD force, and a term proportional to the four-acceleration squared, which we shall call here the ‘‘damping part.’’ The second term of Eq. (1) comes from local coupling of the particle to Ricci curvature. This term preserves the conformal invariance of the SF. The two local terms, namely the ALD and Ricci-coupled terms, constitute the local part of the SF, F_{μ}^{local} . The third term in the SF is the non-local tail term, which involves an integration over the gradient of the retarded Green's function along the entire past world line of the particle.

A number of practical methods have been used in order to find the effects of the tail part of the SF. One such approach avoids the calculation of the SF by applying balance arguments to quantities which are constants of motion in the absence of radiation reaction (i.e., the energy and angular momentum). This method has been successful to quite high relativistic order when the black hole is spherically symmetric, or when the motion is circular or equatorial around a spinning black hole, because in these cases the orbital evolution is uniquely determined by the rate of change of energy and angular momentum, and global conservation laws imply that the fluxes of energy and angular momentum across a distant sphere and down the event horizon of the black hole equal the rate of loss of energy and angular momentum by the particle, respectively. However, this method is useless for cases where there is non-trivial evolution of Carter's constant, which is a non-additive constant of motion in the absence of radiation reaction. This method has in addition two other problems. First, it involves averaging over the orbital period. Thus, when the orbital evolution is fast and the time scale for the orbital evolution becomes comparable with the time scale for the orbital period, this method becomes inaccurate. Second, this method is sensitive to the slowest decaying part of the field at large distances. Consequently, it handles well the dissipative, radiative part of the field, but completely ignores the conservative part of the SF. Although the latter does not cause any net loss of energy and angular momentum, it is still important for the orbital evolution [6]. In fact, even in the absence of dissipation, the conservative SF pushes the particle off the geodesic, and thus causes orbital evolution which may be of practical importance.

A second approach is based on the radiative Green's function [1]. Specifically, one can write the retarded Green's function G^{ret} as the sum of two terms, namely

$$G^{\text{ret}} = \frac{1}{2}(G^{\text{ret}} + G^{\text{adv}}) + \frac{1}{2}(G^{\text{ret}} - G^{\text{adv}}),$$

where G^{adv} is the advanced Green's function. The first term is time symmetric, and consequently does not include the radiative part of the field. Instead, it relates to the non-radiative Coulomb piece of the field, which is the source for the divergence. If one considers then only the second term, namely the radiative Green's function, one can obtain an expression for the field of the charge, which is finite on the world line of the charge, and from this field obtain the SF. Although this approach is very successful in flat spacetime, it suffers from an inherent difficulty in curved spacetime. Specifically, in curved spacetime it is anti-causal: The radiative Green's function includes the advanced Green's function, which in curved spacetime has support inside the future light cone. Consequently, the momentary force on the particle depends in principle, according to the radiative Green's function, on the entire future history of the particle. When the future history is completely known, e.g., an eternal static particle, or eternal circular motion, this approach is expected to yield correct results. However, in general the future history is unknown, and might also be subjected to free will, such that an approach built on the radiative Green's function is unsatisfactory. An approach which is based solely on the causal retarded field is clearly preferable. (In what follows we shall indeed base our calculations on the retarded field.) Also this method ignores the conservative piece of the SF, which is included in the discarded time-symmetric part of G^{ret} .

This approach was used by Gal'tsov to obtain the SF on scalar, electric, and gravitational charges in the spacetime of a Kerr black hole [7]. In particular, Gal'tsov found the SF acting on charges in a uniform circular orbit around a Kerr black hole in the weak field limit, and studied also the forces on static charges in the same limit.

In this Paper we focus on the SF acting on a scalar charge. Although a scalar charge is just a toy model for the more interesting and more realistic gravitational charge (a point mass), it already involves much of the properties of more realistic fields, especially the tail part of the SF. Yet, some of the complications associated with the gravitational SF are not invoked, notably the gauge problem of the SF. Also, the scalar field Green's function has just one component, which simplifies the analysis. Thus, despite its relative simplicity, the problem of the scalar field SF captures the essence of the physics, while avoiding some technical complications. For that reason the scalar field SF has been a very useful toy model.

In what follows we shall compute the self force to linear order in the particle's field [i.e., to order (charge)², and neglect corrections of order (charge)⁴ or higher. We thus assume that the particle's charge is much smaller than the typical length scale for the gravitational field, specifically the black hole's mass.]

For a number of simple cases, e.g., static electric [8] or scalar [9] charges in the spacetime of a Schwarzschild black hole, or a static electric charge on the polar axis of a Kerr black hole [10], the particle's field is known exactly in a closed form. Indeed, these exact solutions were used to find the SF for those cases [11,9,10]. However, in most cases an exact solution for the particle's field is unknown. For sufficiently simple spacetimes, e.g., that of a stationary and axisymmetric black hole, one can decompose the field into Fourier-harmonic modes, which can be obtained relatively easily. Specifically, the individual modes of the field (or the Green's function) satisfy an ordinary differential equation (whereas the field itself satisfies a partial differential equation), which for almost all cases can be solved (at least numerically using standard methods). In addition, it turns out that the individual modes of the field are continuous across the particle's world line and the resulting contributions to the SF of the individual modes are bounded. The divergency arises only at the step of summation over all modes.

This prompted Ori to propose a calculation of the SF effects which is based on the retarded field and on a mode decomposition [12] (in that case the adiabatic, orbit integrated, evolution rate of the constants of motion in Kerr). More recently, Ori proposed to apply that method directly for the calculation of the SF [13]. The greatest challenge, as was already mentioned, lies with an appropriate prescription for the regularization of the mode sum. When the individual modes of the SF are summed naïvely, the result typically diverges. The reason for this divergence is that the modes do not distinguish between the tail and the instantaneous parts of the SF, and contribute to both. [This occurs already in the case of a static scalar or electric charge in flat spacetime, when the position of the charge does not coincide with the center of the coordinates. In that case the contribution to the SF of each mode (after summation over all azimuthal numbers m from $-l$ to l) is independent of the mode number l , and is given by $-q^2/(2r^2)$, q being the (scalar or electric) charge, and r being the position of the charge. Obviously, the sum over modes diverges, and should be removed by a certain regularization prescription.] A mode sum regularization prescription (MSRP), which handles this divergence, and which is an application of the approaches of Mino *et al* [3] and of Quinn and Wald [4], was proposed by Ori [13,14].

Next, we describe the MSRP very succinctly. Further details are presented in Refs. [14,15]. The contribution to the physical SF from the tail part of the Green's function can be decomposed into stationary Teukolsky modes, and then summed over the frequencies ω and the azimuthal numbers m . The tail part of the SF equals then the limit $\epsilon \rightarrow 0^-$ of the sum over all l modes, of the difference between the force sourced by the entire world line (the bare force $^{\text{bare}}f_\mu^l$) and the force sourced by the half-infinite world line to the future of ϵ , where the particle has proper time $\tau = 0$, and $\tau = \epsilon$ is an event along the past ($\tau < 0$) world line. Next, we seek a regularization function h_μ^l which is independent of ϵ , such that the series $\sum_l (^{\text{bare}}f_\mu^l - h_\mu^l)$ converges. Once such a function is found, the regularized tail part of the SF is then given by

$$^{\text{tail}}F_\mu \equiv \lim_{\epsilon \rightarrow 0^-} q^2 \int_{-\infty}^{\tau+\epsilon} d\tau' \nabla_\mu G^{\text{ret}}[z^\alpha(\tau), z^\alpha(\tau')] = \sum_{l=0}^{\infty} (^{\text{bare}}f_\mu^l - h_\mu^l) - d_\mu, \quad (2)$$

where d_μ is a finite valued function. MSRP then shows, from a local integration of the Green's function, that the regularization function $h_\mu^l = a_\mu l + b_\mu + c_\mu l^{-1}$. For several cases, which have already been studied, MSRP yields the values of the functions a_μ, b_μ, c_μ and d_μ analytically. Alternatively, a_μ, b_μ , and c_μ (but not d_μ) can also be found from the large- l behavior of $^{\text{bare}}f_\mu^l$. As $\sum_{l=0}^{\infty} (^{\text{bare}}f_\mu^l - h_\mu^l)$ converges, it is clear that the large- l behavior of $^{\text{bare}}f_\mu^l$ is identical to the large- l behavior of h_μ^l . For more details on MSRP, and in particular on the local integration of the Green's function and the analytical derivation of the MSRP parameters, see Refs. [14,15].

The MSRP has been applied successfully for a number of cases, including the SF on static scalar or electric charges in the spacetime of a Schwarzschild black hole [16], a scalar charge in uniform circular orbit around a Schwarzschild black hole [17], an electric charge in uniform circular motion in Minkowski spacetime [18], a scalar charge which is in radial free fall in the absence of the SF into a Schwarzschild black hole [19], and the SF on static scalar or electric charges inside or outside thin spherical shells [20]. In all these cases the MSRP parameters a_μ, b_μ, c_μ and d_μ were known analytically, and were used in the regularization of the SF (for a list of these parameters see Ref. [21]). The

SF has not been calculated using MSRP yet for cases where the MSRP parameters are unknown. However, there are interesting cases where these parameters have not been found analytically yet. Specifically, they have not been found analytically for any spacetime which is not spherically symmetric. In addition, when the spacetime of a spinning black hole is concerned, their analytical derivation is expected to be considerably more complicated than in the spherically symmetric case: the Green's function is generally time dependent, and for the corresponding wave evolution in the spacetime of a rotating black hole different l modes of the field couple [22], a phenomenon which does not happen in spherically-symmetric spacetimes. As was noted above, the MSRP parameters a_μ , b_μ , and c_μ can be found also by studying the large- l behavior of the individual modes of the bare force. However, without knowledge of the MSRP parameter d_μ any regularized result will not be unambiguous. It appears, then, that a local analysis of the retarded Green's function is unavoidable. Remarkably, it has been found that in all the cases for which the MSRP parameter d_μ is known, it satisfies a simple relation with the local part of the SF. Specifically, it has been shown that in those cases d_μ equals the sum of the ALD force and the Ricci-curvature coupled piece of the SF, i.e., in the case of a scalar charge,

$$d_\mu = \frac{1}{3}q^2 (\ddot{u}_\mu - u_\mu \dot{u}_\alpha \dot{u}^\alpha) + \frac{1}{6}q^2 \mathcal{R}_\mu. \quad (3)$$

It was then conjectured, based on the particular cases for which it was found to be true, that Eq. (3) is generally satisfied, at least for large classes of scalar charges [15,21]. (In Ref. [15] this conjecture applies only to static spherically-symmetric spacetimes. Here we expand the domain of validity of this conjecture to include at the least also the spacetime of a stationary, axially-symmetric black hole. We note that if this conjecture is found to be valid in general, it would not be unexpected.) If this conjecture is true in general, the full SF is given by just

$$F_\mu = \sum_{l=0}^{\infty} (\text{bare } f_\mu^l - h_\mu^l), \quad (4)$$

and all the terms appearing in Eq. (4) can be found by studying the individual modes of the *bare* SF only. [Even if this conjecture is not true in general, Eq. (4) still holds for the classes of cases for which the conjecture is true.]

In this Paper we shall assume that Eq. (3) holds for the case of a static scalar charge in the spacetime of a stationary, axisymmetric black hole, and compute the SF for that case. By a static particle we mean a particle whose Boyer-Lindquist coordinates r , θ , and φ are fixed. Such a particle is static with respect to an undragged, static observer at infinity, but it rotates in the opposite direction to the black hole's spin with respect to a freely falling local observer. This is the first application of MSRP for a spacetime which is not spherically symmetric. In addition, we also use MSRP without prior knowledge of the regularization parameters. By comparing our result for the SF with known results for a Kerr spacetime in the weak field limit (which were obtained using independent methods in Ref. [7]), we shall, in fact, prove the validity of Eq. (3) in that limit. Moreover, by comparing the SF to the flux of angular momentum across the black hole's event horizon, we shall prove that the conjecture (3) is satisfied also for strong fields.

The SF on static charges in the spacetime of black holes was considered in a number of works. The question to be asked then is the following: How is the external force which is needed in order to keep the particle static changed because of the self interaction of the particle? For the case of a static electric charge q in the spacetime of a Schwarzschild black hole of mass M , Smith and Will [23] and Frolov and Zel'nikov [24,11] found that there is a repelling inverse cubic force, which in the frame of a geodesic observer who is momentarily at rest at the position of the charge q is given by

$$F_{\hat{\mu}} = q^2 \frac{M}{r^3} \delta_{\hat{\mu}}^{\hat{r}}, \quad (5)$$

r being the radial Schwarzschild coordinate. This force arises from the tail part of the full SF. The Schwarzschild spacetime is Ricci flat, such that the coupling to Ricci curvature does not contribute to the SF. Also, a static charge in Schwarzschild has zero ALD force, such that the full SF (5) is given by just the tail force.

For the case of a static scalar charge (where the scalar field is minimally-coupled) in Schwarzschild, Wiseman [9] found the interesting result, that the SF is zero (this was found earlier by Zel'nikov and Frolov [11] as a particular case in the more general study of non-minimally coupled scalar fields). Since the SF again is given by just the tail part (as Schwarzschild is Ricci flat, and the ALD force for a static scalar charge is again zero), it turns out that the tail part of the SF for a static scalar charge in Schwarzschild is zero. This vanishing result is in some sense surprising: spacetime is curved in a non-trivial way (which indeed leads to a non-zero tail force for an electric charge), such that an exactly vanishing result for the tail force is not intuitively expected. The following question then arises: Is this result just a consequence of the particular symmetries imposed, which cause the SF to take a zero value? When these

symmetries are relaxed, does the zero result for the SF persist, or do we find a non-zero result? Some aspects of the symmetries were indeed relaxed in subsequent works. In Ref. [17] the scalar charge was not considered anymore as static. Instead, a scalar charge in uniform circular motion around Schwarzschild was considered, and indeed a non-zero tail force, proportional to the angular velocity squared, was found. This tail force vanishes in the limit of zero angular velocity (a static particle). In Ref. [20] the horizon condition was relaxed, and instead the spacetime was taken to be that of a thin spherical shell. Again, a non-zero SF was found, which vanishes in the limit that the radius of the shell approaches its Schwarzschild radius. In these cases, too, the non-zero SF arises from the tail part of the SF. We thus see that the zero result for the SF (and in particular for the tail part of the SF) for the case of a static scalar charge in Schwarzschild is just a degenerate case when more complicated cases are considered, namely, a scalar charge in circular orbit, or a spacetime which is Schwarzschild, but not that of a black hole. Is the result of a zero SF for a static scalar charge outside a Schwarzschild black hole then just an isolated result, or is it a particular case of a wider class of spacetimes?

We shall consider this question for a number of generalizations of the Schwarzschild spacetime. First, we shall add to the black hole electric charge, thus making it a Reissner-Nordström black hole. The Reissner-Nordström spacetime is electrovac, and hence Ricci curved. However, it turns out that the Ricci part of the SF for a static scalar charge vanishes. Also the ALD part of the SF vanishes for that case. What about the tail part of the SF? We find that the tail part of the SF is zero, and so is the full SF. We thus see that by adding electric charge to the black hole, we do not change the zero result for the SF. What about adding angular momentum to the black hole? We study then the SF on a static scalar charge in the spacetime of a Kerr black hole. Spacetime is Ricci flat, but there is a local contribution to the SF from the ALD part of the SF. This contribution is in the $\partial/\partial\varphi$ direction (in Boyer-Lindquist coordinates). (Incidentally, the force which is required in order to hold the particle fixed in the absence of the SF has components only in the $\partial/\partial r$ and $\partial/\partial\theta$ directions.) When we consider the tail part of the SF, we find that it is still zero. Namely, the full SF is given by just the local, ALD part of the SF. We then consider the most general stationary, axially symmetric black hole, by adding both angular momentum and electric charge to the black hole, turning it into a Kerr-Newman black hole. Spacetime now is Ricci curved, and there is a non-zero contribution to the SF from the local Ricci-coupled part of the SF. There is also a contribution from the local ALD part of the SF. However, when we consider the full SF, we find that it equals just the sum of the two local terms, such that the tail part of the SF is again zero.

We thus conclude that the tail part of the SF on a static scalar charge is zero for all stationary, axially symmetric black holes. The zero result in the spacetime of a Schwarzschild black hole turns out then to be just a particular case of a much wider class of spacetimes. In Schwarzschild also the local parts of the SF turn out to be zero, such that the full SF vanishes. However, in more general spacetimes the local terms are non-zero, such that there is a non-zero SF, but the interesting (and the difficult to find) part of the SF is the tail part, and it is zero for all stationary and axisymmetric black holes.

We find that the full, regularized SF on a static scalar charge q in the spacetime of a Kerr-Newman black hole with mass M , spin parameter a , and electric charge Q , is given in Boyer-Lindquist coordinates by

$$F_{\mu}^{\text{SF}} = \frac{1}{3} q^2 a \Delta \sin^2 \theta \frac{M^2 - Q^2}{(\Delta - a^2 \sin^2 \theta)^{5/2} \Sigma^{1/2}} \delta_{\mu}^{\varphi}. \quad (6)$$

Here, q is the scalar charge of the particle, the horizon function $\Delta = r^2 - 2Mr + a^2 + Q^2$, and $\Sigma = r^2 + a^2 \cos^2 \theta$. Equation (6) for the full, regularized SF acting on a static scalar charge in the spacetime of a stationary, axisymmetric black hole is our main result in this Paper.

One striking feature of the SF is that it diverges as the static limit (the outer boundary of the ergosphere) is approached. This situation is much different from that of a static electric charge in Schwarzschild, given by Eq. (5), where the SF is bounded. Also, the ratio of the SF in that case to the regular force which is needed to be applied in order to hold the charge static (in the absence of the SF) tends to zero as the black hole's horizon is approached. Indeed, Smith and Will found that the SF is just a tiny correction, which becomes important only when the theory is no longer expected to be accurate, namely, when quantum effects are needed to be considered [23]. (Smith and Will found that, at its maximum at $r = 3M$, the acceleration due to the SF becomes comparable with the regular acceleration if the Schwarzschild radius of the black hole is smaller than the classical electron radius, and the distance of the electron from the horizon is smaller by two orders of magnitude than the electron's Compton wave length.) In the strong field regime we thus find the SF acting on a scalar charge held static in the spacetime of a stationary, axisymmetric black hole to grow unboundedly as the static limit is approached. Note, that also the regular force which is needed to keep the particle static in the absence of the SF diverges in the same limit. The ratio of the acceleration due to the SF, a^{SF} , and the regular acceleration, a^{reg} , for a Kerr black hole and on the equatorial plane, is found to be

$$\frac{a^{\text{SF}}}{a^{\text{reg}}} = \frac{1}{3} \frac{q^2}{\mu} a \frac{M}{r^2} \frac{1}{r - 2M} \quad (7)$$

where q is the scalar charge of a particle with mass μ , and M and a are the mass and spin of the black hole, correspondingly. By the accelerations of the left hand side we mean the corresponding magnitudes. As the static limit is approached (on the equatorial plane the static limit is at $r = 2M$), this ratio diverges, which signifies that the acceleration due to the SF becomes dominant over the regular acceleration. Also, there is a finite value of r for which the two accelerations become equal. When the charge-to-mass ratio q/μ of the scalar particle is large, the two accelerations become comparable at large distances from the black hole.

The organization of this paper is as follows. In Section II we derive the equations governing the field, and obtain the expression for the SF. In Section III we derive the SF analytically in the weak field regime, and in Section IV we evaluate the SF numerically in the strong field regime. In Section V we derive the SF using the far field and balance arguments, and compare our results with those obtained in Sections III and IV using the near field. Finally, in Section VI we discuss the properties of the SF.

II. FORMULATION

Consider a static scalar charge in the spacetime of a stationary, axially symmetric, black hole, i.e. the Kerr-Newman black hole. By a static particle we mean here that the particle's spatial position is fixed in Boyer-Lindquist coordinates. The background spacetime is described by the Kerr-Newman metric, which in Boyer-Lindquist coordinates assumes the form

$$ds^2 = - \left(1 - \frac{2Mr - Q^2}{\Sigma} \right) dt^2 - \frac{2a \sin^2 \theta (2Mr - Q^2)}{\Sigma} dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\zeta \sin^2 \theta}{\Sigma} d\varphi^2, \quad (8)$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 + a^2 + Q^2 - 2Mr$, and $\zeta = (r^2 + a^2)\Sigma + a^2(2Mr - Q^2) \sin^2 \theta = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta$. We use units in which $G = c = 1$ throughout. Here M , a and Q are respectively the mass, angular momentum per unit mass and electric charge of the black hole.

The linearized field equation for a minimally coupled, massless scalar field Φ is given by

$$\nabla_\mu \nabla^\mu \Phi(x^\alpha) = -4\pi \rho(x^\alpha), \quad (9)$$

where ∇_μ denotes covariant differentiation compatible with the metric (8). The scalar charge density ρ is given by

$$\rho(x^\mu) = q \int_{-\infty}^{\infty} d\tau \frac{\delta^4[x^\mu - z^\mu(\tau)]}{\sqrt{-g}}. \quad (10)$$

Here q is the particle's total scalar charge; τ is the proper time; $g = -\Sigma^2 \sin^2 \theta$ is the metric determinant; and z^μ is the world line of the charge. We assume that the charge is placed at a position $(r_0, \theta_0, \varphi_0)$. To solve Eq. (9), we decompose ρ and Φ into a sum over spherical harmonics $Y_{lm}(\theta, \varphi)$:

$$\rho(r, \theta, \varphi) = q \sqrt{1 - \frac{2Mr_0 - Q^2}{\Sigma_0}} \frac{\delta(r - r_0)}{\Sigma} \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}^*(\theta_0, \varphi_0) Y_{lm}(\theta, \varphi) \quad (11)$$

$$\Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \Phi^{lm}(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \phi^{lm}(r) Y_{lm}(\theta, \varphi), \quad (12)$$

where $\Sigma_0 = r_0^2 + a^2 \cos^2 \theta_0$. Substituting Eqs. (11) and (12) into Eq. (9), we find

$$\Delta \phi_{,rr}^{lm} + 2(r - M) \phi_{,r}^{lm} + \left[\frac{m^2 a^2}{\Delta} - l(l+1) \right] \phi^{lm} = -4\pi q \sqrt{1 - \frac{2Mr_0 - Q^2}{\Sigma_0}} Y_{lm}^*(\theta_0, \varphi_0) \delta(r - r_0), \quad (13)$$

where commas denote partial derivatives. The boundary conditions for Φ are that Φ vanishes as $r \rightarrow \infty$, and is regular on the future event horizon. Regularity of Φ on the future horizon is equivalent to the field being derived from the retarded Green's function. (Regularity on the past event horizon is similarly related to the advanced field.) The solution of Eq. (13) can be written as

$$\phi^{lm}(r) = \frac{\phi_1^{lm}(r_<) \phi_2^{lm}(r_>)}{W_r[\phi_1^{lm}, \phi_2^{lm}](r_0)} S(r_0) Y_{lm}^*(\theta_0, \varphi_0) , \quad (14)$$

where

$$S(r_0) = -\frac{4\pi q}{\Delta_0} \sqrt{1 - \frac{2Mr_0 - Q^2}{\Sigma_0}} . \quad (15)$$

Here $\Delta_0 = r_0^2 + a^2 + Q^2 - 2Mr_0$, $r_> = \max(r, r_0)$, $r_< = \min(r, r_0)$, $W_r[\phi_1^{lm}, \phi_2^{lm}](r_0) = \phi_1^{lm}(r_0)\phi_{2,r}^{lm}(r_0) - \phi_2^{lm}(r_0)\phi_{1,r}^{lm}(r_0)$ is the Wronskian determinant evaluated at $r = r_0$; ϕ_1^{lm} and ϕ_2^{lm} are two independent solutions of the homogeneous equation

$$\Delta\phi_{,rr} + 2(r - M)\phi_{,r} + \left[\frac{m^2 a^2}{\Delta} - l(l+1) \right] \phi = 0 , \quad (16)$$

with ϕ_2^{lm} satisfying the boundary condition at infinity, and ϕ_1^{lm} chosen appropriately to make Φ^{lm} regular on the future event horizon.

Equation (16) has three regular singular points at r_+ , r_- , and at infinity, where $r_{\pm} = M \pm \sqrt{M^2 - Q^2 - a^2}$ are the outer and inner horizons of the black hole. We next move the regular singular points of Eq. (16) to $\pm 1, \infty$. This is done by the transformation

$$z(r) = \frac{2r - r_+ - r_-}{r_+ - r_-} . \quad (17)$$

Equation (16) then becomes

$$(1 - z^2)\phi_{,zz} - 2z\phi_{,z} + \left[l(l+1) - \frac{\mu^2}{(1 - z^2)} \right] \phi = 0 , \quad (18)$$

which is the associated Legendre equation. Here the degree μ is purely imaginary and is given by

$$\mu = im\gamma \quad , \quad \gamma = \frac{a}{\sqrt{M^2 - a^2 - Q^2}} . \quad (19)$$

Two linearly independent solutions are $P_l^\mu(z)$ and $Q_l^\mu(z)$, the associated Legendre function of the first and second kinds [25], respectively. The functions $P_l^\mu(z)$ and $Q_l^\mu(z)$ can be expressed in terms of hypergeometric functions:

$$P_l^\mu(z) = \frac{1}{\Gamma(1 - \mu)} \left(\frac{z+1}{z-1} \right)^{\mu/2} {}_2F_1 \left(-l, l+1; 1 - \mu; \frac{1-z}{2} \right) \quad (20)$$

$$Q_l^\mu(z) = e^{i\mu\pi} 2^{-l-1} \sqrt{\pi} \frac{\Gamma(l + \mu + 1)}{\Gamma(l + \frac{3}{2})} z^{-l-\mu-1} (z^2 - 1)^{\mu/2} {}_2F_1 \left(1 + \frac{l+\mu}{2}, \frac{l+\mu+1}{2}; l + \frac{3}{2}; \frac{1}{z^2} \right) . \quad (21)$$

The hypergeometric function has the Gauss series representation

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n \quad |z| < 1 , \quad (22)$$

where Pochhammer's symbol is defined to be

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1) \cdots (x+n-1) \quad (n \geq 1) . \quad (23)$$

Note that the hypergeometric function which appears in the $P_l^\mu(z)$ expression (20) is just a polynomial of order l , and the series expansion (22) is valid even though the magnitude of the argument is greater than one.

As $r \rightarrow \infty$ ($z \rightarrow \infty$), $P_l^\mu(z) \sim z^l$ and $Q_l^\mu(z) \sim z^{-l-1}$. In order for Φ to vanish at infinity, we must have $\phi_2^{lm}(z) = Q_l^\mu(z)$ up to an arbitrary multiplicative factor [Note, from Eq. (14), that the value of ϕ^{lm} does not change if ϕ_1^{lm} or ϕ_2^{lm} is multiplied by a factor independent of r].

To determine $\phi_1^{lm}(z)$, we first consider the case $m = 0$. The function $\Phi^{l0}(r, \theta, \varphi) = \phi(r) Y_{l0}(\theta, \varphi)$ is independent of φ , such that the singularity of the coordinate φ on the horizon does not complicate the analysis. The general solution

of the homogeneous equation is the linear combinations of the Legendre functions $P_l(z)$ and $Q_l(z)$. On the horizon $z = 1$, Q_l diverges but P_l remains finite, so we have $\phi_1^{l0}(z) = P_l(z)$ up to a multiplicative factor. When $m \neq 0$, we find it more convenient to expand the general solution as a linear combination of $P_l^\mu(z)$ and $P_l^{-\mu}(z)$ instead of $P_l^\mu(z)$ and $Q_l^\mu(z)$. Note that $P_l^{-\mu}(z)$ is a solution of (18) since the equation is invariant when μ is changed to $-\mu$. The two functions behave as

$$P_l^{\pm\mu}(z) \sim \frac{1}{\Gamma(1 \mp \mu)} \left(\frac{z+1}{z-1} \right)^{\pm\mu/2} = \frac{1}{\Gamma(1 \mp \mu)} \left(\frac{r-r_-}{r-r_+} \right)^{\pm i\mu a/(r_+-r_-)} \quad (24)$$

as $r \rightarrow r_+$. Both solutions oscillate near the horizon because of the coordinate singularity of φ there. To remove this coordinate singularity, we consider the coordinate $\tilde{\varphi}(\varphi, r)$ defined such that

$$d\tilde{\varphi} = d\varphi + \frac{a}{\Delta} dr .$$

Upon integration, we have

$$\tilde{\varphi} = \varphi + \int \frac{a}{\Delta} dr = \varphi + \frac{a}{r_+ - r_-} \ln \left(\frac{r-r_+}{r-r_-} \right) \quad (25)$$

plus an integration constant which we set equal to zero without loss of generality (it only relates to the origin of the coordinate $\tilde{\varphi}$.) Hence

$$e^{im\tilde{\varphi}} = \left(\frac{r-r_-}{r-r_+} \right)^{i\mu a/(r_+-r_-)} e^{im\varphi} .$$

In terms of the $(r, \theta, \tilde{\varphi})$ coordinates (the ingoing Kerr-Newman coordinates), the two functions $P_l^\mu(z)Y_{lm}(\theta, \varphi)$ and $P_l^{-\mu}(z)Y_{lm}(\theta, \varphi)$ become

$$\frac{1}{\Gamma(1-\mu)} Y_{lm}(\theta, \tilde{\varphi}) \left(\frac{r-r_-}{r-r_+} \right)^{2i\mu a/(r_+-r_-)} \quad \text{and} \quad \frac{1}{\Gamma(1+\mu)} Y_{lm}(\theta, \tilde{\varphi})$$

as $r \rightarrow r_+$ ($z \rightarrow 1$), respectively. (When regularity of Φ on the past event horizon is required, the appropriate coordinate transformation is given by $d\tilde{\varphi} = d\varphi - \frac{a}{\Delta} dr$.) The first expression still oscillates near the horizon while the second one is regular. Combining with the $m = 0$ case, we conclude that $\phi_1^{lm}(z) = P_l^{-\mu}(z)$ up to a multiplicative factor.

Note that each mode of the potential $\Phi^{lm}(r, \theta, \varphi) = \phi^{lm}(r)Y_{lm}(\theta, \varphi)$ is complex in general. However, it is easy to show that $\Phi^{l,-m} = (\Phi^{lm})^*$, such that the m -sum in Eq. (12) is real and so is the scalar field Φ .

The modes of the bare SF, f_μ^l , are given by

$$f_\mu^l = \sum_{m=-l}^l q \nabla_\mu \Phi^{lm}(r_0, \theta_0, \varphi_0) . \quad (26)$$

It follows from Eqs. (12) and (14) that $\Phi_{,\theta}^{lm}$ and $\Phi_{,\varphi}^{lm}$ are continuous at the position of the charge, but $\Phi_{,r}^{lm}$ is not. Hence f_r^l is not uniquely defined. However, the MSRP guarantees that the regularized SF does not depend on which derivative $[\Phi_{,r}^{lm}(r_0^+), \Phi_{,r}^{lm}(r_0^-)]$ (where r_0^\pm are the one sided limits of $r \rightarrow r_0$ from above or below, correspondingly) or the average of the two] we use [15]. In practice, we use the average derivative, i.e., we define

$$f_r^l = \frac{1}{2} \sum_{m=-l}^l q [\Phi_{,r}^{lm}(r_0^+) + \Phi_{,r}^{lm}(r_0^-)] . \quad (27)$$

Explicitly, we find that

$$f_r^l(r_0, \theta_0, \varphi_0) = \frac{1}{2} \sum_{m=-l}^l \frac{qS(r_0)}{W_r[\phi_1^{lm}, \phi_2^{lm}](r_0)} [\phi_1^{lm}(r_0)\phi_{2,r}^{lm}(r_0) + \phi_{1,r}^{lm}(r_0)\phi_2^{lm}(r_0)] Y_{lm}^*(\theta_0, \varphi_0) Y_{lm}(\theta_0, \varphi_0) , \quad (28)$$

$$f_{\theta}^l(r_0, \theta_0, \varphi_0) = \sum_{m=-l}^l \frac{qS(r_0)}{W_r[\phi_1^{lm}, \phi_2^{lm}](r_0)} \phi_1^{lm}(r_0) \phi_2^{lm}(r_0) Y_{lm}^*(\theta_0, \varphi_0) \partial_{\theta} Y_{lm}(\theta_0, \varphi_0), \quad (29)$$

and

$$f_{\varphi}^l(r_0, \theta_0, \varphi_0) = \sum_{m=-l}^l \frac{qS(r_0)}{W_r[\phi_1^{lm}, \phi_2^{lm}](r_0)} \phi_1^{lm}(r_0) \phi_2^{lm}(r_0) Y_{lm}^*(\theta_0, \varphi_0) \partial_{\varphi} Y_{lm}(\theta_0, \varphi_0). \quad (30)$$

Typically, $\sum_l f_{\mu}^l$ diverges when summed naïvely. We then use MSRP to regularize the SF as outlined in Section I. Recall that the MSRP parameters a_{μ} , b_{μ} , and c_{μ} can be determined by two independent methods. Specifically, they can be found by either (i) study of the large- l behavior of the individual modes of the bare SF, or by (ii) a local analysis of the Green function. In this paper, we shall study the regularization parameters using the former method. (It is hard to apply the latter method here because of the following reason. Any time-dependent evolution of the wave equation in the spacetime of a rotating black hole has to handle mode couplings, and the Green's function is obviously time dependent [22].)

We shall carry out the regularization procedure analytically in Section III to study the SF in the weak field regime, and then numerically in Section IV in the strong field regime.

III. COMPUTATION OF SELF-FORCE IN THE WEAK FIELD REGIME

In this Section, we consider the case for which the scalar charge is far away from the black hole, i.e. $r_0 \gg M$ [$z_0 = (2r_0 - r_+ - r_-)/(r_+ - r_-) \gg 1$]. We can thus expand f_{μ}^l in Eq. (26) in powers of r_0^{-1} .

The large z_0 expansion of ϕ_1^{lm} and ϕ_2^{lm} can be carried out by using Eqs. (20) and (21). Specifically, we choose

$$\phi_1^{lm}(z) = \frac{2^l l!}{(2l)!} \Gamma(1 + \mu + l) P_l^{-\mu}(z) = \frac{2^l l!}{(2l)!} \frac{\Gamma(1 + \mu + l)}{\Gamma(1 + \mu)} \left(\frac{z+1}{z-1} \right)^{-\mu/2} {}_2F_1 \left(-l, l+1; 1 + \mu; \frac{1-z}{2} \right) \quad (31)$$

$$\phi_2^{lm}(z) = \frac{e^{-i\mu\pi} 2^{l+1}}{\sqrt{\pi}} \frac{\Gamma(l + \frac{3}{2})}{\Gamma(l + \mu + 1)} Q_l^{\mu}(z) = z^{-l-\mu-1} (z^2 - 1)^{\mu/2} {}_2F_1 \left(1 + \frac{l+\mu}{2}, \frac{l+\mu+1}{2}; l + \frac{3}{2}; \frac{1}{z^2} \right). \quad (32)$$

It can be shown, from Eq. (22), that the above expressions are equivalent to

$$\phi_1^{lm}(z) = z^l \left(1 + \frac{1}{z} \right)^{-\mu/2} \left(1 - \frac{1}{z} \right)^{l+\mu/2} \sum_{n=0}^l \frac{(-l)_n (-\mu-l)_n}{n! (-2l)_n} \left(\frac{2}{1-z} \right)^n \quad (33)$$

$$\phi_2^{lm}(z) = \frac{1}{z^{l+1}} \left(1 - \frac{1}{z^2} \right)^{\mu/2} \sum_{n=0}^{\infty} \frac{\left(1 + \frac{l+\mu}{2} \right)_n \left(\frac{l+\mu+1}{2} \right)_n}{n! \left(l + \frac{3}{2} \right)_n} \frac{1}{z^{2n}}. \quad (34)$$

Up to this point, no approximation has been made, but Eqs. (14), (33), (34) and (26) give us a convenient way to expand f_{μ}^l in powers of r_0^{-1} . Hereafter, we evaluate all quantities at the position of the particle. To simplify the notation we shall assume that the scalar charge is placed at (r, θ, φ) and drop all the subscripts “0”.

We first consider the expansions for f_r^l and f_{θ}^l . We shall compute f_r^l up to the order r^{-7} and f_{θ}^l up to the order r^{-6} . Straightforward calculations give

$$f_r^l = \frac{qS(r)}{2} \left\{ \frac{X_l(0, \theta)}{2l+1} + \frac{2[l(l+1)X_l(0, \theta) + 3\gamma^2 X_l(2, \theta)]}{(2l-1)(2l+1)(2l+3)} \frac{1}{z^2} + \frac{6}{(2l-3)(2l-1)(2l+1)(2l+3)(2l+5)} \times \right. \\ \left. [l(l+1)(l^2+l-3)X_l(0, \theta) + 2(3l^2+3l-5)\gamma^2 X_l(2, \theta) + 5\gamma^4 X_l(4, \theta)] \frac{1}{z^4} + O\left(\frac{1}{z^6}\right) \right\} \quad (35)$$

$$f_{\theta}^l = -q\sqrt{M^2 - a^2 - Q^2} S(r) \left\{ \frac{2\gamma^2 \xi_l(2, \theta)}{(2l-1)(2l+1)(2l+3)} \frac{1}{z} + \frac{2[3\gamma^4 \xi_l(4, \theta) + 2l(l+1)\gamma^2 \xi_l(2, \theta)]}{(2l-3)(2l-1)(2l+1)(2l+3)(2l+5)} \frac{1}{z^3} + O\left(\frac{1}{z^5}\right) \right\}, \quad (36)$$

where

$$X_l(p, \theta) = \sum_{m=-l}^l m^p Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) \quad (37)$$

$$\xi_l(p, \theta) = \sum_{m=-l}^l m^p Y_{lm}^*(\theta, \varphi) \partial_{\theta} Y_{lm}(\theta, \varphi). \quad (38)$$

In writing Eqs. (35) and (36), we have used the results that $\xi_l(0, \theta) = 0$ and $X_l(p, \theta) = \xi_l(p, \theta) = 0$ if p is a positive odd integer. We leave the detailed calculations of the two functions $X_l(p, \theta)$ and $\xi_l(p, \theta)$ to Appendix A. Substituting the expressions of X_l and ξ_l from Appendix A [Eqs. (A17)–(A22)], we obtain

$$f_r^l = q \frac{S(r)}{8\pi} \left\{ 1 + \frac{2l(l+1)}{(2l-1)(2l+3)} \left[1 + \frac{3\gamma^2}{2} \sin^2 \theta \right] \frac{1}{z^2} + \frac{6l(l+1)}{(2l-3)(2l-1)(2l+3)(2l+5)} \times \right. \\ \left. \left[(l^2 + l - 3) + (3l^2 + 3l - 5)\gamma^2 \sin^2 \theta + \frac{5}{2}\gamma^4 \sin^2 \theta + \frac{15}{8}\gamma^4(l-1)(l+2) \sin^4 \theta \right] \frac{1}{z^4} + O\left(\frac{1}{z^6}\right) \right\} \quad (39)$$

$$f_\theta^l = -q \frac{S(r)}{4\pi} \sqrt{M^2 - a^2 - Q^2} \gamma^2 \sin \theta \cos \theta \left\{ \frac{l(l+1)}{(2l-1)(2l+3)} \frac{1}{z} + \frac{2l(l+1)}{(2l-3)(2l-1)(2l+3)(2l+5)} \times \right. \\ \left. \left[l(l+1) + \frac{3}{2}\gamma^2 + \frac{9}{4}(l-1)(l+2)\gamma^2 \sin^2 \theta \right] \frac{1}{z^3} + O\left(\frac{1}{z^5}\right) \right\} . \quad (40)$$

For large values of l we find

$$f_r^{l \gg 1} = \frac{qS(r)}{8\pi} \left[1 + \frac{1}{2} \left(1 + \frac{3\gamma^2}{2} \sin^2 \theta \right) \frac{1}{z^2} + \frac{3}{8} \left(1 + 3\gamma^2 \sin^2 \theta + \frac{15}{8}\gamma^4 \sin^4 \theta \right) \frac{1}{z^4} + O\left(\frac{1}{z^6}\right) \right] + O\left(\frac{1}{l^2}\right) \quad (41)$$

$$f_\theta^{l \gg 1} = -\frac{qS(r)}{16\pi} \sqrt{M^2 - a^2 - Q^2} \gamma^2 \sin \theta \cos \theta \left[\frac{1}{z} + \frac{1}{2} \left(1 + \frac{9}{4}\gamma^2 \sin^2 \theta \right) \frac{1}{z^3} + O\left(\frac{1}{z^5}\right) \right] + O\left(\frac{1}{l^2}\right) . \quad (42)$$

Hence we find that the MSRP parameters $a_r = c_r = 0$ up to the order of r^{-7} (i.e., any deviation of a_r or c_r from zero is of order r^{-8} or higher) [Note that $S(r)$ gives an extra factor r^{-2}], $a_\theta = c_\theta = 0$ up to the order of r^{-6} (i.e., any deviation of a_θ or c_θ from zero is of order r^{-7} or higher), and b_r and b_θ are given from Eqs. (41) and (42):

$$b_r = \frac{qS(r)}{8\pi} \left[1 + \frac{1}{2} \left(1 + \frac{3\gamma^2}{2} \sin^2 \theta \right) \frac{1}{z^2} + \frac{3}{8} \left(1 + 3\gamma^2 \sin^2 \theta + \frac{15}{8}\gamma^4 \sin^4 \theta \right) \frac{1}{z^4} + O\left(\frac{1}{z^6}\right) \right] \quad (43)$$

and

$$b_\theta = -\frac{qS(r)}{16\pi} \sqrt{M^2 - a^2 - Q^2} \gamma^2 \sin \theta \cos \theta \left[\frac{1}{z} + \frac{1}{2} \left(1 + \frac{9}{4}\gamma^2 \sin^2 \theta \right) \frac{1}{z^3} + O\left(\frac{1}{z^5}\right) \right] . \quad (44)$$

[For the case for which the particle is on the polar axis of the Kerr-Newman black hole we find that

$$b_r^{\text{axis}} = -\frac{q^2}{2r^2} \frac{1 - M/r}{1 - 2M/r + (a^2 + Q^2)/r^2} \left(1 + \frac{a^2}{r^2} \right)^{-1/2} \delta_\mu^r . \quad (45)$$

We have checked the latter expression numerically in the strong field regime, and found complete agreement.] Assuming the conjecture (4), the r -component of the regularized SF is then calculated by subtracting Eq. (43) from Eq. (39) and then summing over l . The θ -component is evaluated similarly from Eqs. (44) and (40). The results are

$$F_r = \sum_{l=0}^{\infty} (f_r^l - b_r) = 0 + O\left(\frac{1}{r^8}\right) \quad (46)$$

$$F_\theta = \sum_{l=0}^{\infty} (f_\theta^l - b_\theta) = 0 + O\left(\frac{1}{r^7}\right) , \quad (47)$$

where we have used the fact that for any integer k

$$\sum_{l=0}^{\infty} \left[\frac{1}{2(l-k)+1} - \frac{1}{2(l+k)+1} \right] = 0 . \quad (48)$$

Hence we conclude that any non-zero orthonormal r and θ components of the SF ($F_{\hat{r}}$ and $F_{\hat{\theta}}$) are of order r^{-8} or higher. In the next section, we present strong numerical evidence to suggest that F_r and F_θ are actually zero wherever the location of the charge.

The φ -component of the SF can also be computed in the same way. However, we find a better method to do the calculation, which is described in detail in Appendix B. We find that

$$f_\varphi^l = -\frac{qS(r)}{z^{2l}} a \left(1 - \frac{1}{z^2}\right) \sum_{m=-l}^l m^2 \left[\prod_{j=1}^l (m^2 \gamma^2 + j^2) \right] \left[\frac{K_l^\mu(z)}{(2l+1)!!} \right]^2 Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi), \quad (49)$$

where $(2l+1)!! = 1 \cdot 3 \cdot 5 \cdots (2l+1)$ and

$$K_l^\mu(z) = \left(1 - \frac{1}{z^2}\right)^{\mu/2} {}_2F_1 \left(1 + \frac{l+\mu}{2}, \frac{l+\mu+1}{2}; l + \frac{3}{2}; \frac{1}{z^2}\right). \quad (50)$$

It follows from Eq. (49) that the leading term of f_φ^l is of order r^{-2l-2} . So when we make an asymptotic expansion by keeping terms up to r^{-N} , only finite number of terms with $l \leq (N-2)/2$ contribute. In other words, up to the order r^{-N} , $f_\varphi^l = 0$ when $l > (N-2)/2$, such that the MSRP parameters $a_\varphi = b_\varphi = c_\varphi = 0$. In the next Section, we show numerically that f_φ^l decreases exponentially with increasing l , which also suggests that no regularization is needed to compute F_φ . Expanding Eq. (49) in powers of r^{-1} , we find that

$$F_\varphi = \sum_{l=0}^{\infty} f_\varphi^l = \frac{q^2}{3} a \sin^2 \theta \frac{M^2 - Q^2}{r^4} \left\{ 1 + 3 \frac{M}{r} + \frac{1}{2} [3(5M^2 - Q^2) + 2a^2(1 - 3 \cos^2 \theta)] \frac{1}{r^2} + O\left(\frac{1}{r^3}\right) \right\}. \quad (51)$$

The leading order term of this expansion agrees with the result given in Ref. [7] for the case where $Q = 0$. This agreement implies that we use here the correct parameter d_φ , and that the conjecture (3) holds for the case studied here. Note that this conclusion is valid only for the weak-field regime of a Kerr spacetime. In the next sections we bring evidence that this conjecture holds also for cases for which $Q \neq 0$, and also in strong fields.

IV. COMPUTATION OF THE SELF FORCE IN THE STRONG FIELD REGIME

Next, we compute the SF in the strong-field regime. To do so, we use the general expressions (28)–(30) for the modes of the field, and compute them numerically. In order to find the regularized SF, we shall use MSRP. As discussed above, we determine the MSRP parameters a_μ , b_μ , and c_μ by studying the large- l behavior of the individual modes of the bare SF. We shall, however, check our numerical results for the regularization parameters in the spherically symmetric limit, where the regularization parameters are known analytically [15]. This is done in subsection IV A, where we study the case of a Reissner-Nordström black hole. Then, in subsection IV B we consider the case of a Kerr black hole, and in subsection IV C we consider the case of a Kerr-Newman black hole. We shall also compare our results for the strong field with the weak-field approximation of Section III.

A. Reissner-Nordström

Our main goal in this Paper is to study the SF on a static scalar charge in the spacetime of a rotating black hole. We shall apply our computation also for the spherically symmetric, electrically charge, Reissner-Nordström black hole for two reasons. First, as we already noted, the SF acting on a static scalar charge in the spacetime of a Reissner-Nordström black hole has not been calculated yet. Our results in this subsection are thus new. Second, it will allow us to check our numerical code for a case where the MSRP parameters are known analytically. (Our code does not assume spherical symmetry, thus its computation of the angular dependence of functions is non-trivial.)

It is clear from symmetry considerations that all the azimuthal components vanish, such that we check below only the radial component of the SF. In Reissner-Nordström, it was found analytically in Ref. [15] that for a static scalar charge

$$b_r^{\text{RN}} = -\frac{q^2}{2r^2} \frac{1 - M/r}{1 - 2M/r + Q^2/r^2}, \quad (52)$$

and $a_r^{\text{RN}} = 0 = c_r^{\text{RN}}$. Also, $d_r^{\text{RN}} = 0$. Figure 1 displays our results for a Reissner-Nordström black hole of mass $M = 1$ and electric charge $Q = 0.8M$, for a particle at $r_0 = 4M$. In Fig. 1(A) we present the modes of the covariant radial component of the bare SF, f_r^l , for a Reissner-Nordström spacetime. These modes appear to approach a constant value in the large- l limit. In order to check whether this limiting value coincides with b_r^{RN} , we plot in Fig. 1(B) the difference between f_r^l and b_r^{RN} as a function of l . As for Reissner-Nordström it was shown that the MSRP parameter $d_r^{\text{RN}} = 0$, and also the ALD and Ricci-curvature parts of the SF vanish, the conjecture (3) holds. Hence this difference

is just the modes of the regularized full SF. We find this difference to scale like l^{-2} for large values of l . This result is in full accord with the analytical results for the MSRP parameters. We also compute the regularized full SF. In Fig. 1(C) we plot $F_r^l = \sum_{j=0}^l (f_r^j - b_r^{\text{RN}})$ as a function of l . We find that F_r^l behaves like l^{-1} in the large- l limit, such that $F_r^l \rightarrow 0$ as $l \rightarrow \infty$. We infer that $F_r \equiv F_r^{l \rightarrow \infty} = 0$ for a static scalar charge in Reissner-Nordström. Similar results were obtained for other choices of the parameters. We infer that both the tail, non-local part of the SF, and the local two terms of the SF vanish separately for a static scalar charge outside a Reissner-Nordström black hole. This generalizes the known result of a zero SF on a static scalar charge in Schwarzschild to Reissner-Nordström.

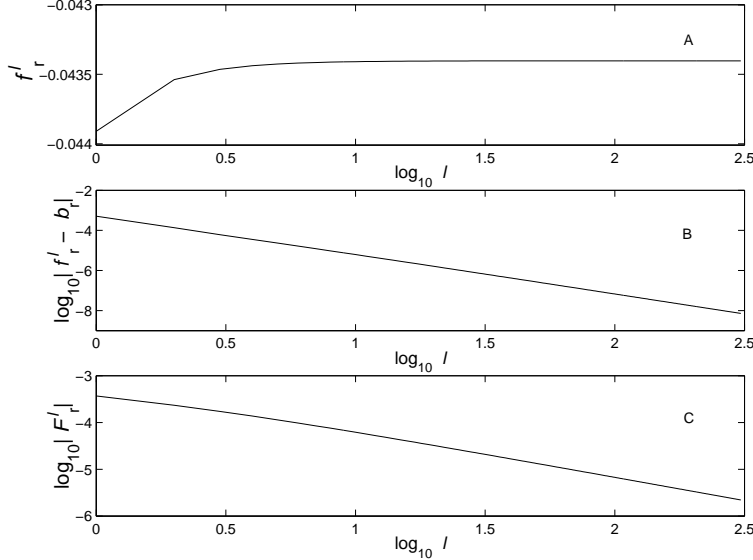


FIG. 1. Self force on a static scalar charge in Reissner-Nordström. Upper panel (A): f_r^l as a function of l . Middle panel (B): The difference $f_r^l - b_r^{\text{RN}}$ as a function of l . Lower panel (C): F_r^l as a function of l . The data presented here correspond to the parameters: $r = 4M$, $a = 0$, $Q = 0.8M$, and $M = 1$. For the actual computation we took $\theta = \pi/4$.

B. Kerr

When the black hole is endowed with non-zero spin, we no longer have analytical results for the MSRP parameters a_μ , b_μ and c_μ . As noted above, these parameters can be found, however, from the large- l behavior of the modes of the bare SF. Nevertheless, the MSRP parameter d_μ cannot be found by studying the modes of the bare SF alone, and a local analysis of the Green's function is necessary in order to determine it. Recently, a conjecture about the MSRP parameter d_μ was formulated [15,21]. According to this conjecture, the MSRP parameter d_μ equals the sum of the two local terms in the full regularized SF, as given by Eq. (3). In the Kerr spacetime, which is Ricci flat, d_μ equals then just the ALD force. In the following we shall use this conjecture. The ALD part of the SF is given by

$$F_\mu^{\text{ALD}} = \frac{1}{3} q^2 a M^2 \Delta \frac{\sin^2 \theta}{(\Delta - a^2 \sin^2 \theta)^{5/2} \Sigma^{1/2}} \delta_\mu^\varphi. \quad (53)$$

We thus conjecture that $d_\mu = F_\mu^{\text{local}}$. Also, because spacetime is Ricci flat, $F_\mu^{\text{local}} = F_\mu^{\text{ALD}}$.

Figure 2 displays the behavior of the individual modes of the bare SF for a static scalar charge in the spacetime of a Kerr black hole (i.e., the scalar charge has fixed Boyer-Lindquist coordinates r, θ and φ .) We choose here the parameters $r = 2.2M$, $\theta = \pi/4$, $a = 0.2M$, $Q = 0$, and $M = 1$, but similar results were found also for other choices. We find that f_r^l and f_θ^l approach constants as $l \rightarrow \infty$, and that the difference between two consecutive modes, $f_r^l - f_r^{l-1}$ and $f_\theta^l - f_\theta^{l-1}$, behave like l^{-3} in the large- l limit. This behavior implies that the MSRP parameters a_r^{Kerr} , a_θ^{Kerr} , c_r^{Kerr} , and c_θ^{Kerr} vanish. (Non-zero a_μ^{Kerr} implies linear growth of the modes with the mode number l , and non-zero c_μ^{Kerr} implies that the difference between two consecutive modes behaves like l^{-2} . Moreover, non-zero c_μ^{Kerr} threatens the applicability of MSRP, as it implies a divergent d_μ^{Kerr} [15].) Recall that we have defined a_μ in the averaged sense. The corresponding “one sided” values are not zero, in general. We also find that f_φ^l decays exponentially with l for large values of l , which suggests that $a_\varphi^{\text{Kerr}} = 0 = c_\varphi^{\text{Kerr}}$, and, in addition, also $b_\varphi^{\text{Kerr}} = 0$. The vanishing of b_φ^{Kerr} is in accord with our results in the weak-field expansions in Section III.

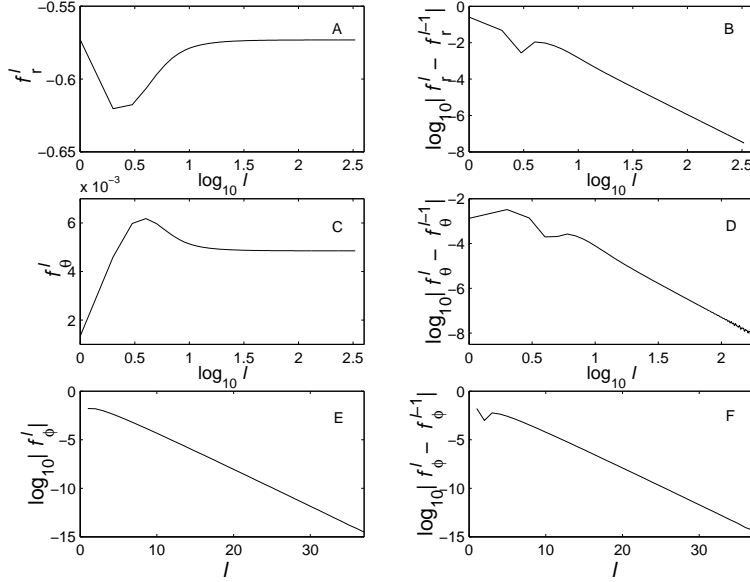


FIG. 2. Behavior of the individual modes of the self force for a static scalar charge in the spacetime of a Kerr black hole. Panels (A), (C), and (E): The r , θ , and φ covariant components of the individual modes of the self force, respectively (i.e., f_r^l , f_θ^l , and f_φ^l), as functions of the mode number l . Panels (B), (D), and (F): The r , θ , and φ covariant components of the difference between two consecutive modes of the self force, respectively (i.e., $f_r^l - f_r^{l-1}$, $f_\theta^l - f_\theta^{l-1}$, and $f_\varphi^l - f_\varphi^{l-1}$), as functions of the mode number l . The parameters for the data presented here are: $r = 2.2M$, $\theta = \pi/4$, $a = 0.2M$, $Q = 0$, and $M = 1$.

In order to compute the regularized SF we have to confront the difficulty of not having exact expressions for the MSRP parameter b_μ^{Kerr} . However, b_μ^{Kerr} is nothing but the limit as $l \rightarrow \infty$ of f_μ^l . We can thus approximate b_μ^{Kerr} by studying the large- l behavior of f_μ^l , and extrapolate to infinite value of the mode number (e.g., through Richardson extrapolation). In practice, we can approximate b_μ^{Kerr} by simply taking the mode of the bare SF with a mode number L much larger than the mode l up to which we sum over the modes to obtain the full, summed-over-modes, SF. That is, we compute the regularized SF according to

$$F_\mu = \sum_{j=0}^{\infty} (f_\mu^j - f_\mu^\infty) \equiv \sum_{j=0}^l (f_\mu^j - f_\mu^\infty) + \mathcal{R}_\mu^{l+1} \quad (54)$$

$$\equiv \sum_{j=0}^l (f_\mu^j - f_\mu^L) + \mathcal{R}_\mu^{l+1} + \mathcal{E}_\mu^L. \quad (55)$$

The functions \mathcal{R}_μ^{l+1} and \mathcal{E}_μ^L are defined by Eqs. (54) and (55), respectively. Now, by definition

$$\begin{aligned} \mathcal{R}_\mu^{l+1} &= \sum_{j=l+1}^{\infty} (f_\mu^j - f_\mu^\infty) \\ &= \sum_{j=l+1}^{\infty} \left[\frac{x_\mu}{j^2} + O(j^{-3}) \right], \end{aligned} \quad (56)$$

where x_μ is the coefficient of the l^{-2} term in the l^{-1} expansion of f_μ^l . In evaluating \mathcal{R}_μ^{l+1} we shall drop the $O(j^{-3})$ term in Eq. (56). This introduces an error which reduces like l^{-2} as l grows. By definition, the function $\mathcal{E}_\mu^L = \sum_{j=0}^l (f_\mu^j - f_\mu^\infty)$, and as inside the sum we just have a j -independent expression, we find that $\mathcal{E}_\mu^L = (l+1)(f_\mu^L - f_\mu^\infty)$. Taking $L \gg l \gg 1$, we approximate F_μ in practice by

$$F_\mu^l \approx \sum_{j=0}^l (f_\mu^j - f_\mu^L). \quad (57)$$

The error associated with this approximation has two contributions, i.e., \mathcal{R}_μ^{l+1} and \mathcal{E}_μ^L . We evaluate $\mathcal{R}_\mu^{l+1} \approx x_\mu \psi^{(1)}(l+1) \approx x_\mu/l$, where $\psi^{(1)}(z)$ is the trigamma function. We also evaluated $\mathcal{E}_\mu^L \approx lx_\mu/L^2$. Note, that x_μ can be evaluated

from the difference between two consecutive modes, i.e., $x_\mu \approx l^3(f_\mu^{l-1} - f_\mu^l)/2$ (when l is large enough). We find that $\mathcal{E}_\mu^L/\mathcal{R}_\mu^{l+1} \approx (l/L)^2$, such that when $L \gg l$ the overall error is dominated by \mathcal{R}_μ^{l+1} .

Figure 3 shows the behavior of the regularized SF. We choose here $l = 330$ and $L = 1000$. In Fig. 3(A) and 3(B) we show $f_r^l - b_r$ and F_r^l as functions of l , respectively, and in Fig. 3(C) and 3(D) we show $f_\theta^l - b_\theta$ and F_θ^l as functions of l , respectively. As expected, we find both $f_r^l - b_r$ [Fig. 3(A)] and $f_\theta^l - b_\theta$ [Fig. 3(C)] to behave like l^{-2} for large values of l . We also find that F_r^l [Fig. 3(B)] and F_θ^l [Fig. 3(D)] scale like l^{-1} for large values of l . When extrapolated, we conclude that as $l \rightarrow \infty$, $F_r^l \rightarrow F_r = 0$ and $F_\theta^l \rightarrow F_\theta = 0$.

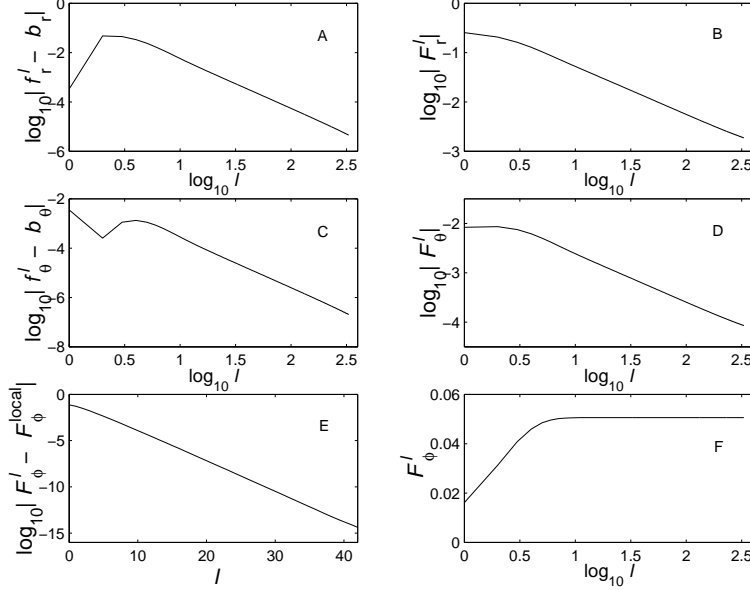


FIG. 3. The regularized SF for a static scalar charge in the spacetime of a Kerr black hole. Panels (A) and (C): The r and θ covariant components of the individual modes of the regularized self force, respectively, (i.e., $f_r^l - b_r$ and $f_\theta^l - b_\theta$) as functions of the mode number l . Panels (B) and (D): $F_r^l = \sum_{j=0}^l (f_r^j - b_r)$ and $F_\theta^l = \sum_{j=0}^l (f_\theta^j - b_\theta)$, respectively, as functions of l . Panel (E): The difference between $F_\varphi^l = \sum_{j=0}^l (f_\varphi^j - b_\varphi)$ and the local SF, F_φ^{local} , which is given by Eq. (53), as a function of l . Panel (F): F_ϕ^l as a function of l . The (unknown) values of b_r and b_θ were approximated by their respective values at $L = 1000$, i.e., by $f_r^{L=1000}$ and $f_\theta^{L=1000}$, correspondingly. The parameters for the data presented here are: $r = 2.2M$, $\theta = \pi/4$, $a = 0.2M$, $Q = 0$, and $M = 1$.

We can approximate F_r^l and F_θ^l even better by including approximate values for \mathcal{R}_μ^{l+1} and \mathcal{E}_μ^L . This is done in Fig. 4. In Fig. 4(A) we show F_r^l twice: without the inclusion of \mathcal{R}_r^{l+1} and \mathcal{E}_r^L [same as in Fig. 3(B)] and with their inclusion. We find that for large values of l , the latter behaves like l^{-2} (the former scales like l^{-1}). Similarly, in Fig. 4(B) we show F_θ^l twice: without the inclusion of \mathcal{R}_θ^{l+1} and \mathcal{E}_θ^L [same as in Fig. 3(D)] and with their inclusion. Again, we find that for large values of l , the latter behaves like l^{-2} (the former scales like l^{-1}). Recall that we are using here only approximated values for \mathcal{R}_μ^{l+1} and \mathcal{E}_μ^L . For this reason, we do not expect their inclusion to yield an exact zero result for the r and θ components of the SF. However, they do eliminate the leading order term in F_r^l and F_θ^l , such that instead of a leading l^{-1} behavior we find an l^{-2} behavior. When this behavior is extrapolated to $l \rightarrow \infty$, we again infer that $F_r^l \rightarrow F_r = 0$ and $F_\theta^l \rightarrow F_\theta = 0$.

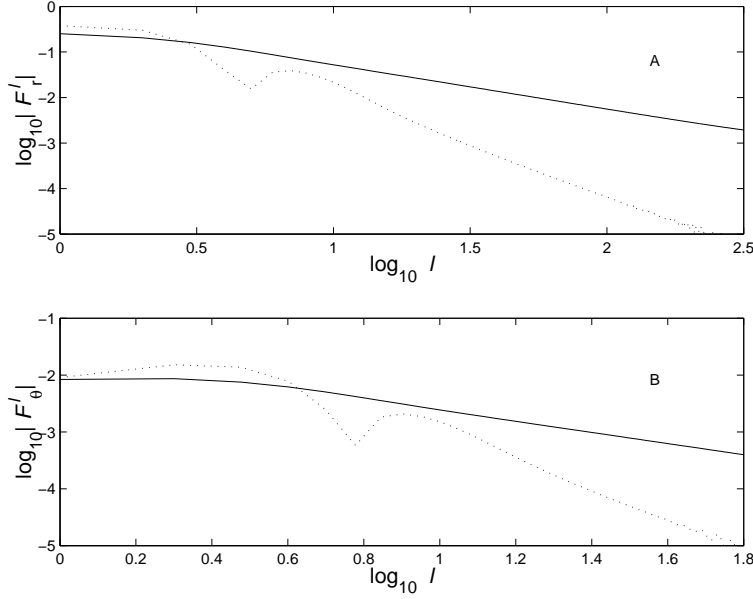


FIG. 4. The regularized SF for a static scalar charge in the spacetime of Kerr. Upper panel (A): F_r^l without the inclusion of \mathcal{R}_r^{l+1} and \mathcal{E}_r^L [same as in Fig. 3(B)] (solid line) and with their inclusion (dotted line). Lower panel (B): F_θ^l without the inclusion of \mathcal{R}_θ^{l+1} and \mathcal{E}_θ^L [same as in Fig. 3(D)] (solid line) and with their inclusion (dotted line).

In Fig. 3(F) we show F_φ^l as a function of l , and in Fig. 3(E) we show the difference between F_φ^l and F_φ^{local} , which is given by Eq. (53). We find that F_φ^l approaches a non-zero value as $l \rightarrow \infty$. We also find that the difference between F_φ^l and F_φ^{local} decays exponentially in l for large- l values. We infer that as $l \rightarrow \infty$, $F_\varphi^l \rightarrow F_\varphi = F_\varphi^{\text{local}}$. We find similar results also for other choices of the parameters.

When our results for the SF are combined, we find that the tail part of the regularized SF vanishes. That is, the tail part is given by

$$F_\mu^{\text{tail}} = q^2 \int_{-\infty}^{\tau^-} d\tau \nabla_\mu G^{\text{tail}} = \sum_{j=0}^{\infty} (f_\mu^j - b_\mu) - d_\mu. \quad (58)$$

According to the conjecture (3), $d_\mu = F_\mu^{\text{local}}$. We find numerically that $\sum_{j=0}^{\infty} (f_\mu^j - b_\mu) = F_\mu^{\text{local}}$. Consequently, we infer that $F_\mu^{\text{tail}} = 0$.

C. Kerr-Newman

Next, we endow the black hole with an electric charge Q in addition to its spin parameter a . In this case, too, we do not have the analytical results for the MSRP parameters a_μ , b_μ and c_μ . As in the Kerr case above, these parameters can be obtained from the large- l behavior of the modes of the bare SF. Again, we shall use the conjecture (3) regarding the MSRP parameter d_μ , according to which d_μ equals the sum of the two local terms in the full, regularized SF. As the Kerr-Newman spacetime has non-vanishing Ricci curvature, both terms contribute. We find that for a static scalar charge outside a Kerr-Newman black hole

$$F_\mu^{\text{ALD}} = \frac{1}{3} q^2 a \Delta \sin^2 \theta \frac{M^2 \Sigma + Q^2 (Q^2 - 2Mr)}{(\Delta - a^2 \sin^2 \theta)^{5/2} \Sigma^{3/2}} \delta_\mu^\varphi, \quad (59)$$

and

$$F_\mu^{\text{Ricci}} = -\frac{1}{3} q^2 a \Delta \sin^2 \theta \frac{Q^2}{(\Delta - a^2 \sin^2 \theta)^{3/2} \Sigma^{3/2}} \delta_\mu^\varphi, \quad (60)$$

such that the total local piece of the SF is given by

$$F_\mu^{\text{local}} = \frac{1}{3} q^2 a \Delta \sin^2 \theta \frac{M^2 - Q^2}{(\Delta - a^2 \sin^2 \theta)^{5/2} \Sigma^{1/2}} \delta_\mu^\varphi. \quad (61)$$

First, we study the behavior of the individual modes f_μ^l of the bare SF, which we compute numerically from Eqs. (28)–(30). Figure 5 shows the individual modes, f_μ^l , and the difference between two consecutive modes, $f_\mu^l - f_\mu^{l-1}$, as functions of the mode number l for $\mu = r, \theta, \varphi$. In Fig. 5 we present the first 336 modes (i.e., $l = 0, \dots, 335$) for the parameters $r = 2.2M$, $\theta = \pi/4$, $a = 0.1M$, $Q = 0.1M$, and $M = 1$, but similar behavior was found also for other choices for the values of the parameters. We find that the r and θ components of the modes approach a non-zero limiting value in the large- l limit, as the difference between two consecutive modes behaves like l^{-3} for large values of l . This behavior of the individual modes implies that the MSRP parameters a_r , a_θ , c_r , and c_θ vanish. (Recall that we are using here the “averaged” value for a_r and a_θ . The “one-sided” values are, in general, non-zero.) (Note, that a non-zero value for c_r or c_θ implies that the difference between two consecutive modes should scale like l^{-2} .) However, the MSRP parameters b_r and b_θ are non-zero. These parameters correspond to the $l \rightarrow \infty$ limit of the individual modes f_r^l and f_θ^l , respectively (recall that a_r and a_θ vanish). The φ component of the individual modes of the SF drops off exponentially for large values of l , and we infer that $a_\varphi = 0$, $b_\varphi = 0$, and $c_\varphi = 0$. This is in agreement with the weak-field expansion of Section III.

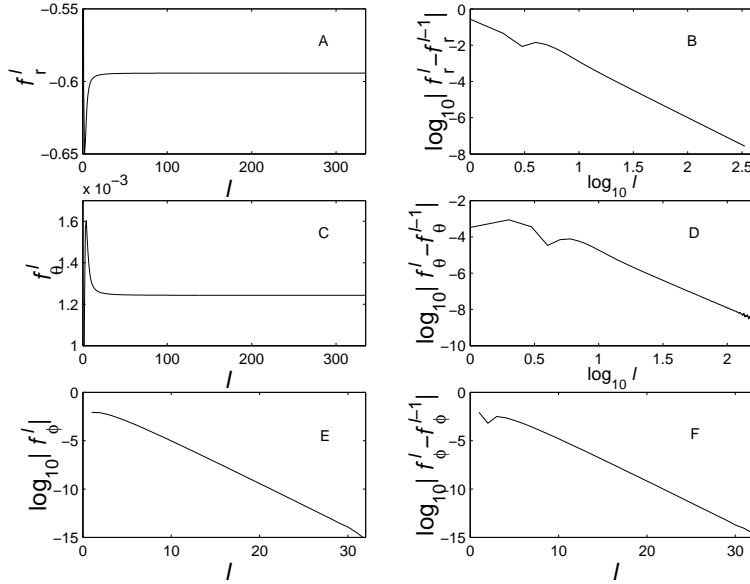


FIG. 5. Behavior of the individual modes of the self force for a static scalar charge in the spacetime of a Kerr-Newman black hole. Panels (A), (C), and (E): The r , θ , and φ covariant components of the individual modes of the self force, respectively (i.e., f_r^l , f_θ^l , and f_φ^l), as functions of the mode number l . Panels (B), (D), and (F): The r , θ , and φ covariant components of the difference between two consecutive modes of the self force, respectively (i.e., $f_r^l - f_r^{l-1}$, $f_\theta^l - f_\theta^{l-1}$, and $f_\varphi^l - f_\varphi^{l-1}$), as functions of the mode number l . The parameters for the data presented here are: $r = 2.2M$, $\theta = \pi/4$, $a = 0.1M$, $Q = 0.1M$, and $M = 1$.

Next we compare our results for the modes here, with our asymptotic expansions in Section III. Figure 6 displays the relative difference of the asymptotic values for the modes of the bare SF [given by Eq. (39) for the r component and by Eq. (40) for the θ component] and the full values of the modes of the bare SF [which we compute numerically from Eqs. (28) and (29)] as functions of r (in units of $r_{\text{SL}} = M + \sqrt{M^2 - a^2 \cos^2 \theta}$, the r value at the static limit) for different values of the mode number l . At large distances we find both the r and θ components to agree with the asymptotic expansion at the right orders. Near the static limit, they of course disagree. Note, however, that even very close to the static limit the r component of the modes of the asymptotic expansion does not deviate from their full-expression counterparts by much more than 10%. We find similar results also for other choices of the parameters.

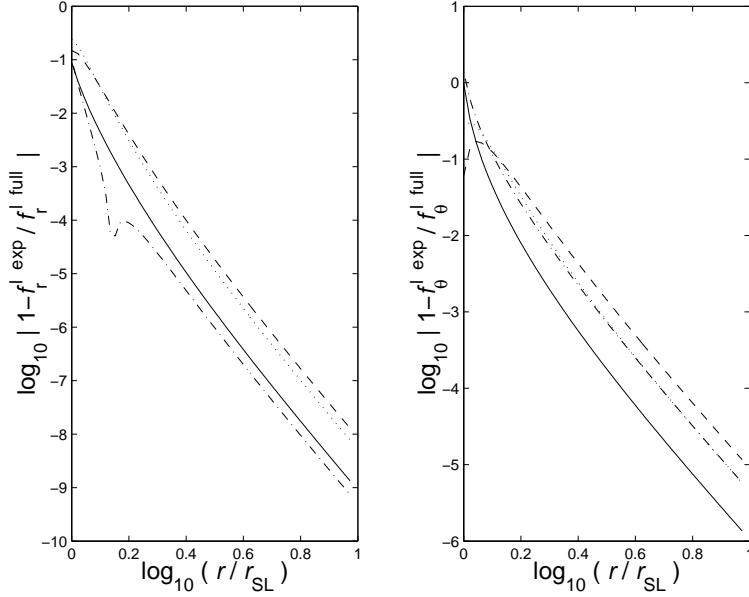


FIG. 6. Comparison of the modes of the bare force as given by the asymptotic expansions [Eq. (39) for the r component and Eq. (40) for the θ component] and the full expressions which we compute numerically from Eqs. (28) and (29), respectively. We show the relative difference of the two expressions as a function of r/r_{SL} for a number of mode numbers l . Left panel: the r component. Right panel: the θ component. In both panels we show the modes: $l = 1$ (solid line), $l = 2$ (dash-dotted line), $l = 3$ (dashed line), and $l = 4$ (dotted line). The data here are shown for the following values for the parameters: $M = 1$, $a = 0.6M$, $Q = 0.4M$, and $\theta = \pi/4$.

Figure 7 displays the behavior of the regularized SF. We choose here again $l = 330$ and $L = 1000$. In Fig. 7(A) and 7(B) we show $f_r^l - b_r$ and F_r^l as functions of l , respectively, and in In Fig. 7(C) and 7(D) we show $f_\theta^l - b_\theta$ and F_θ^l as functions of l , respectively. Similar to the Kerr case, we find both $f_r^l - b_r$ [Fig. 7(A)] and $f_\theta^l - b_\theta$ [Fig. 7(C)] to behave like l^{-2} for large values of l . We also find that F_r^l [Fig. 7(B)] and F_θ^l [Fig. 7(D)] scale like l^{-1} for large values of l . When extrapolated, we conclude that as $l \rightarrow \infty$, $F_r^l \rightarrow F_r = 0$ and $F_\theta^l \rightarrow F_\theta = 0$.

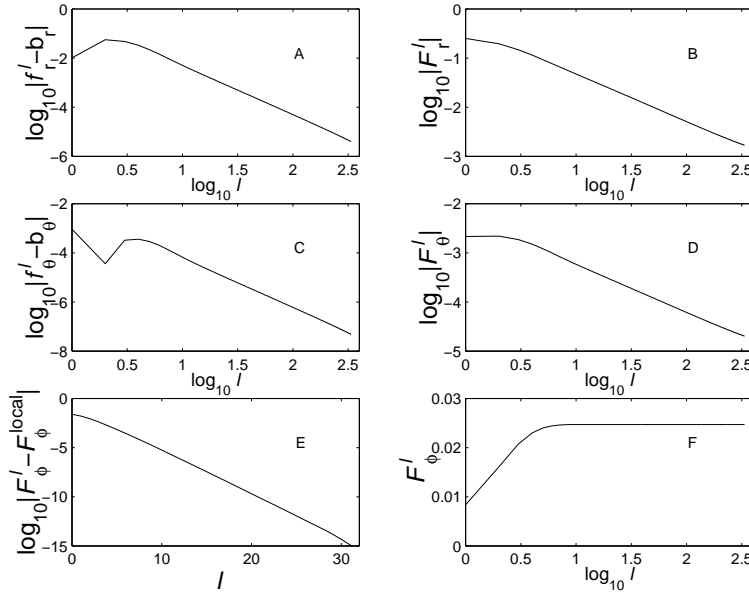


FIG. 7. The regularized SF for a static scalar charge in the spacetime of a Kerr-Newman black hole. Panels (A) and (C): The r and θ covariant components of the individual modes of the regularized self force, respectively, (i.e., $f_r^l - b_r$ and $f_\theta^l - b_\theta$) as functions of the mode number l . Panels (B) and (D): $F_r^l = \sum_{j=0}^l (f_r^j - b_r)$ and $F_\theta^l = \sum_{j=0}^l (f_\theta^j - b_\theta)$, respectively, as functions of l . Panel (E): The difference between $F_\varphi^l = \sum_{j=0}^l (f_\varphi^j - b_\varphi)$ and the local SF, F_φ^{local} , as a function of l . Panel (F): F_φ^l as a function of l . The (unknown) values of b_r and b_θ were approximated by their respective values at $L = 1000$, i.e., by $f_r^{L=1000}$ and $f_\theta^{L=1000}$, correspondingly. The parameters for the data presented here are: $r = 2.2M$, $\theta = \pi/4$, $a = 0.1M$, $Q = 0.1M$, and $M = 1$.

In a similar way to our analysis of the case of a Kerr spacetime, we can approximate F_r^l and F_θ^l even better by including \mathcal{R}_μ^{l+1} and \mathcal{E}_μ^L . This is done in Fig. 8. In Fig. 8(A) we show F_r^l twice: without the inclusion of \mathcal{R}_r^{l+1} and \mathcal{E}_r^L [same as in Fig. 7(B)] and with their inclusion. We again find that for large values of l , the latter behaves like l^{-2} (the former scales like l^{-1}). Similarly, in Fig. 8(B) we show F_θ^l twice: without the inclusion of \mathcal{R}_θ^{l+1} and \mathcal{E}_θ^L [same as in Fig. 7(D)] and with their inclusion. Again, we find that for large values of l , the latter behaves like l^{-2} (the former scales like l^{-1}). Recall that we are using here only approximated values for \mathcal{R}_μ^{l+1} and \mathcal{E}_μ^L . For this reason, we do not expect their inclusion to yield an exact zero result for the r and θ components of the SF. However, like in the Kerr case, they do eliminate the leading order term in F_r^l and F_θ^l , such that instead of a leading l^{-1} behavior we find an l^{-2} behavior. When this behavior is extrapolated to $l \rightarrow \infty$, we again infer that $F_r^l \rightarrow F_r = 0$ and $F_\theta^l \rightarrow F_\theta = 0$.

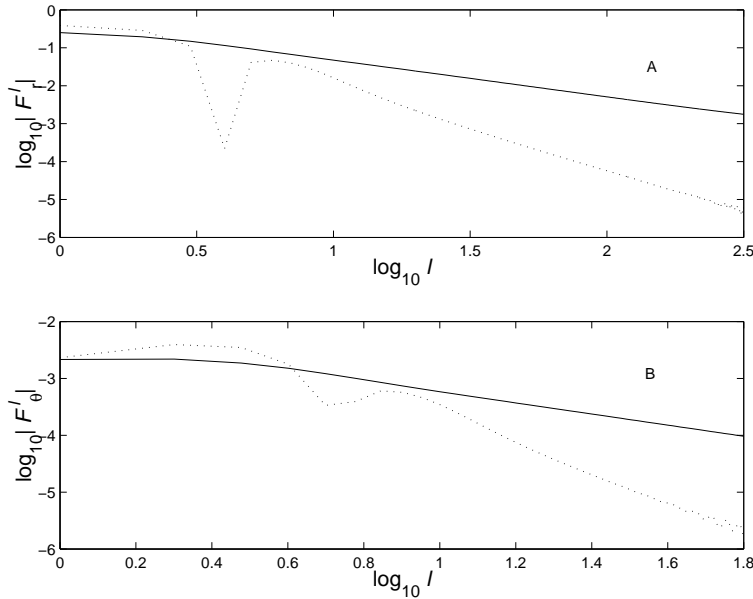


FIG. 8. The regularized SF for a static scalar charge in the spacetime of a Kerr-Newman black hole. Upper panel (A): F_r^l without the inclusion of \mathcal{R}_r^{l+1} and \mathcal{E}_r^L [same as in Fig. 7(B)] (solid line) and with their inclusion (dotted line). Lower panel (B): F_θ^l without the inclusion of \mathcal{R}_θ^{l+1} and \mathcal{E}_θ^L [same as in Fig. 7(D)] (solid line) and with their inclusion (dotted line).

In Fig. 7(F) we show F_φ^l as a function of l , and in Fig. 7(E) we show the difference between F_φ^l and F_φ^{local} , which is given by Eq. (61). We find that F_φ^l approaches a non-zero value as $l \rightarrow \infty$. We also find that the difference between F_φ^l and F_φ^{local} decays exponentially in l for large- l values. We infer that as $l \rightarrow \infty$, $F_\varphi^l \rightarrow F_\varphi = F_\varphi^{\text{local}}$. We find similar results also for other choices of the parameters.

As in the case of a Kerr spacetime, when our results for the SF are combined, we find that the tail part of the regularized SF vanishes.

V. FAR FIELD COMPUTATION OF THE SELF FORCE

In Sections III and IV we computed the SF by using the field (and its gradient) evaluated on the particle's world line, i.e., by using the near field. In this Section, we shall compute the SF using the far field (evaluated asymptotically at infinity and at the black hole's event horizon), and demonstrate the compatibility of the two approaches. For simplicity, we shall restrict our considerations in this Section to the case of a Kerr black hole. Specifically, we shall show that the covariant t and φ components of the SF (F_t and F_φ , respectively) in the case of a Kerr black hole

can also be inferred from balance arguments pertaining to the global conservation of energy and angular momentum. Specifically, we shall deduce F_t and F_φ by calculating the fluxes of energy and angular momentum, associated with the charge's scalar field, flowing out to infinity and down the black hole's event horizon. We shall show that the results of these far-field calculations agree with the near-field calculations we have performed in Sections III and IV. In fact, by showing this agreement we demonstrate the applicability of the MSRP, and the validity of the conjecture (3) to the problem of interest.

Since we need to evaluate the scalar field near the horizon and at infinity, we place the particle at $(r_0, \theta_0, \varphi_0)$ in this Section to avoid confusion.

The rate of change of the particle's four-momentum due to the SF, F_μ , is given by

$$\frac{dp_\mu}{d\tau} = F_\mu , \quad (62)$$

where τ is the proper time. In particular, we have

$$F_\varphi = \frac{dp_\varphi}{d\tau} = \frac{1}{\sqrt{-g_{tt}}} \frac{dL}{dt} , \quad (63)$$

where $L \equiv p_\varphi$ is the angular momentum of the particle along the black hole's rotation axis, and we have used the fact that $dt/d\tau = 1/\sqrt{-g_{tt}}$ (recall that the charge is static). The sum of the rate of change of the particle's angular momentum (i.e., dL/dt), the total amount of angular momentum (per unit time) flowing out to infinity (\mathcal{F}_∞^L) and down the black hole's event horizon ($\mathcal{F}_{\text{hole}}^L$) must be zero, if global angular momentum were to be conserved. Hence,

$$-dL/dt = \mathcal{F}_\infty^L + \mathcal{F}_{\text{hole}}^L . \quad (64)$$

The value of \mathcal{F}_∞^L is given by (see Chapter 5 of Ref. [28])

$$\mathcal{F}_\infty^L = \lim_{r \rightarrow \infty} \int T_\mu{}^r \xi_{(\varphi)}^\mu r^2 d\Omega , \quad (65)$$

where $\xi_{(\varphi)} = \partial/\partial\varphi$ is the axial Killing vector, $d\Omega = \sin\theta d\theta d\varphi$, and the stress-energy tensor $T_{\mu\nu}$ associated with the scalar field Φ is given by

$$T_{\mu\nu} = \frac{1}{4\pi} \left(\Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta} \right) . \quad (66)$$

The total angular momentum flowing down the event horizon per unit time, $\mathcal{F}_{\text{hole}}^L$, is given by [29,30]

$$\mathcal{F}_{\text{hole}}^L = \lim_{r \rightarrow r_+} \int [-2Mr_+ T_{\mu\nu} \xi_{(\varphi)}^\mu l_{HH}^\nu] d\Omega , \quad (67)$$

where l_{HH}^μ is one of the basis vectors of the Hawking-Hartle tetrad. It is an outgoing tetrad which is made well behaved on the future event horizon [31,30,29]. The components of l_{HH}^ν , in Boyer-Lindquist coordinates (t, r, θ, φ) , are given by

$$l_{HH}^\mu = \left[\frac{1}{2}, \frac{\Delta}{2(r^2 + a^2)}, 0, \frac{a}{2(r^2 + a^2)} \right] . \quad (68)$$

Note that although some of the components of $T_{\mu\nu}$, in Boyer-Lindquist coordinates, diverge on the event horizon, $\mathcal{F}_{\text{hole}}^L$ remains finite. In fact, the divergence is due entirely to the coordinate singularity. However, $\mathcal{F}_{\text{hole}}^L$, being a scalar, is independent of the choice of coordinates.

The asymptotic expressions of the scalar field Φ at infinity and on the horizon can be deduced from Eqs. (12), (14), (B6), (B7), (B11), (B8), (15), (17) and (24). The results are

$$\Phi(r, \theta, \varphi) \rightarrow \sqrt{1 - \frac{2Mr_0}{\Sigma_0}} \frac{q}{r} \quad r \rightarrow \infty \quad (69)$$

$$\Phi(r, \theta, \varphi) \rightarrow \sum_{l=0}^{\infty} \sum_{m=-l}^l Z_{lm} e^{-ik_m r_*} Y_{lm}(\theta, \varphi) \quad r \rightarrow r_+ , \quad (70)$$

where r_* is defined by $dr_*/dr = (r^2 + a^2)/\Delta$, $k_m = -m\omega_+ \equiv -ma/(2Mr_+)$, and Z_{lm} is given by

$$Z_{lm} = C_{lm} \frac{\sqrt{M^2 - a^2} S(r_0) (1 - z_0^2)}{z_0^{l+1} (2l+1)!!} \left[\prod_{j=1}^l (j + im\gamma) \right] K_l^\mu(z_0) Y_{lm}^*(\theta_0, \phi_0) . \quad (71)$$

Here C_{lm} are complex constants of unit modulus, i.e. $|C_{lm}| = 1$. Substituting Eq. (69) into Eqs. (66) and (65), we find that $\mathcal{F}_\infty^L = 0$. This is expected since the particle is static relative to static observers at infinity, so no radiation is emitted. The rate of change of the particle's angular momentum is then given by

$$\frac{dL}{dt} = -\mathcal{F}_{\text{hole}}^L = \lim_{r \rightarrow r_+} \left[2Mr_+ \int T_{\varphi\nu} l_{HH}^\nu d\Omega \right] . \quad (72)$$

Straightforward calculations yield

$$\mathcal{F}_{\text{hole}}^L = - \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{m^2 a}{4\pi} |Z_{lm}|^2 \quad (73)$$

$$= - \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{a(M^2 - a^2) S^2(r_0)}{4\pi z_0^{2l-2}} \left(1 - \frac{1}{z_0^2}\right)^2 m^2 \left[\prod_{j=1}^l (m^2 \gamma^2 + j^2) \right] \left[\frac{K_l^\mu(z_0)}{(2l+1)!!} \right]^2 Y_{lm}^*(\theta_0, \varphi_0) Y_{lm}(\theta_0, \varphi_0) . \quad (74)$$

The covariant φ -component of the SF, F_φ , is then calculated by Eq. (63). Using Eqs. (15) and (17), we finally obtain

$$F_\varphi = - \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{qaS(r_0)}{z_0^{2l}} \left(1 - \frac{1}{z_0^2}\right) m^2 \left[\prod_{j=1}^l (m^2 \gamma^2 + j^2) \right] \left[\frac{K_l^\mu(z_0)}{(2l+1)!!} \right]^2 Y_{lm}^*(\theta_0, \varphi_0) Y_{lm}(\theta_0, \varphi_0) , \quad (75)$$

which agrees with Eq. (49) (Recall that the MSRP parameters $a_\varphi = b_\varphi = c_\varphi = 0$).

Similarly, we can deduce the covariant time component of the SF, F_t , by the energy balance argument. We have

$$F_t = \frac{dp_t}{d\tau} = - \frac{1}{\sqrt{-g_{tt}}} \frac{dE}{dt} , \quad (76)$$

where $E \equiv -p_t$ is the energy of the particle. The sum of the rate of change of the particle's energy (i.e., dE/dt), the total amount of the energy (per unit time) associated with the scalar field flowing out to infinity (\mathcal{F}_∞^E) and down the horizon ($\mathcal{F}_{\text{hole}}^E$) vanishes because of global conservation of energy, i.e.,

$$- dE/dt = \mathcal{F}_\infty^E + \mathcal{F}_{\text{hole}}^E . \quad (77)$$

The relevant formulae are [28–30]:

$$\mathcal{F}_\infty^E = \lim_{r \rightarrow \infty} \int -T_\mu{}^r \xi_{(t)}^\mu r^2 d\Omega \quad (78)$$

$$\mathcal{F}_{\text{hole}}^E = \lim_{r \rightarrow r_+} \int 2Mr_+ T_{\mu\nu} \xi_{(t)}^\mu l_{HH}^\nu d\Omega , \quad (79)$$

where $\xi_{(t)} = \partial/\partial t$ is a Killing vector. Straightforward calculations yield $\mathcal{F}_\infty^E = 0 = \mathcal{F}_{\text{hole}}^E$. Hence $dE/dt = 0 = F_t$, as expected.

VI. PROPERTIES OF THE SELF FORCE

From our numerical study we infer that the full, regularized SF on a static scalar charge in Kerr-Newman is given by

$$F_\mu^{\text{SF}} = \frac{1}{3} q^2 a \Delta \sin^2 \theta \frac{M^2 - Q^2}{(\Delta - a^2 \sin^2 \theta)^{5/2} \Sigma^{1/2}} \delta_\mu^\varphi . \quad (80)$$

We note the following properties of this result.

- The SF vanishes as $a \rightarrow 0$, as expected. In that limit the black hole becomes Reissner-Nordström (or Schwarzschild in the lack of electric charge), for which the SF vanishes, as we found in Section IV A.
- When $t \rightarrow -t$ the SF reverses its sign. [Notice that the SF has the form of $a \times (\text{even powers of } a)$.] This is indeed expected, because under time reversal the black hole reverses its spin, and rotates in the opposite direction. This change of sign under time reversal implies that this SF is dissipative.
- The SF diverges as $\Delta - a^2 \sin^2 \theta \rightarrow 0$. This is not surprising, as $\Delta - a^2 \sin^2 \theta = 0$ defines the static limit, beyond which (inside the ergosphere) no timelike static trajectories exist. Consequently, our problem of finding the SF on a static particle becomes ill posed beyond the static limit.
- The SF is only in the $\partial/\partial\varphi$ direction. The addition to the regular force which is needed to keep the particle in its static position is, therefore, orthogonal to the regular force (which has components only in the $\partial/\partial r$ and $\partial/\partial\theta$ directions).
- Denoting the magnitude of the acceleration of the static particle in the absence of the SF by a^{reg} , and the magnitude of the acceleration due only to the SF by a^{SF} , we find the ratio of the two accelerations to be given by

$$\frac{a^{\text{SF}}}{a^{\text{reg}}} = \frac{1}{3} \frac{q^2}{\mu} a \sin \theta \frac{\Sigma^{1/2} \Delta^{1/2}}{\Delta - a^2 \sin^2 \theta} \left\{ \Delta [M(r^2 - a^2 \cos^2 \theta) - rQ^2]^2 + \frac{1}{4} a^4 (2Mr - Q^2)^2 \sin^2 2\theta \right\}^{-1/2}, \quad (81)$$

where μ is the mass of the scalar charge. Clearly, this ratio diverges as the static limit is approached. In this sense, the SF is not a tiny correction for the regular external force which is exerted in order to keep the particle fixed: the SF becomes dominant over the regular acceleration when the particle is sufficiently close to the static limit. The origin of the divergence of this ratio is in the “damping part” of the ALD force. The “Schott part” of the ALD force diverges too approaching the static limit, but at a slower rate than the “damping part”. The part of the SF which couples to Ricci curvature diverges at the same rate as a^{reg} approaching the static limit. The leading order divergence in Eq. (81) comes then from the “damping part” of the ALD force. This can be readily understood by the following argument. The “damping part” of the ALD force diverges like a^{reg^2} , such that its ratio to a^{reg} is expected to diverge like a^{reg} . Indeed, a^{reg} diverges like $(\Delta - a^2 \sin^2 \theta)^{-1}$ approaching the static limit.

The scalar field theory does not restrict the charge-to-mass ratio of a scalar particle, q/μ . (For other field theories we find the charge-to-mass ratios to span many orders of magnitude: it is unity for a gravitational charge, but 2×10^{21} for an electron.) We can thus view q/μ as a free parameter in Eq. (81). Recall, however, that q/M is assumed to be a small quantity. As $r \rightarrow \infty$ the ratio $a^{\text{SF}}/a^{\text{reg}} \rightarrow 0$. Thus, there is a value of r (outside the static limit) where $a^{\text{SF}} = a^{\text{reg}}$. (Recall, however, that the SF acceleration is in the $\partial/\partial\varphi$ direction, while the regular acceleration is in the $\partial/\partial r$ and $\partial/\partial\theta$ directions.) Specializing now to the equatorial plane in Kerr, we find that a^{SF} equals a^{reg} at

$$r = \frac{2}{3}M + \frac{2^{-\frac{1}{3}}}{3}M^{\frac{2}{3}}\mu^{-\frac{1}{3}} \left[27a\frac{q^2}{M} + 16\mu M + 3\sqrt{3a\frac{q^2}{M} \left(27a\frac{q^2}{M} + 32\mu M \right)} \right]^{\frac{1}{3}} + \frac{2^{\frac{7}{3}}}{3}\mu^{\frac{1}{3}}M^{\frac{4}{3}} \left[27a\frac{q^2}{M} + 16\mu M + 3\sqrt{3a\frac{q^2}{M} \left(27a\frac{q^2}{M} + 32\mu M \right)} \right]^{-\frac{1}{3}}. \quad (82)$$

For small values of $aq^2/(\mu M^2)$, this value of r can be expanded in powers of $aq^2/(\mu M^2)$. We find that

$$r = 2M \left\{ 1 + \frac{1}{24} \left(\frac{q}{\mu} \right) \left(\frac{q}{M} \right) \left(\frac{a}{M} \right) - \frac{1}{288} \left(\frac{q}{\mu} \right)^2 \left(\frac{q}{M} \right)^2 \left(\frac{a}{M} \right)^2 + O \left[\left(\frac{q}{\mu} \right)^3 \left(\frac{q}{M} \right)^3 \left(\frac{a}{M} \right)^3 \right] \right\},$$

such that for small $aq^2/(\mu M^2)$ the two accelerations become comparable only very close to the static limit. Figure 9 displays this value for r vs. the free parameter q/μ for three values of q/M . Keeping a/M and q/M fixed, we find that at small values of q/μ the two accelerations become comparable only very close to the static limit. For large values of q/μ they become equal at values of r which scale like $(q/\mu)^{1/3}$. The change in the behavior occurs near $q/\mu \approx M^2/(aq)$. Recalling that $q/M \ll 1$ and that $a/M < 1$, we find that at the change

of behavior in Fig. 9, $q/\mu \gg 1$. In order to have $r \gg M$ (the distance from the black hole at which the two accelerations become comparable is very large) we should thus require $q/\mu \gg M^2/(aq)$.

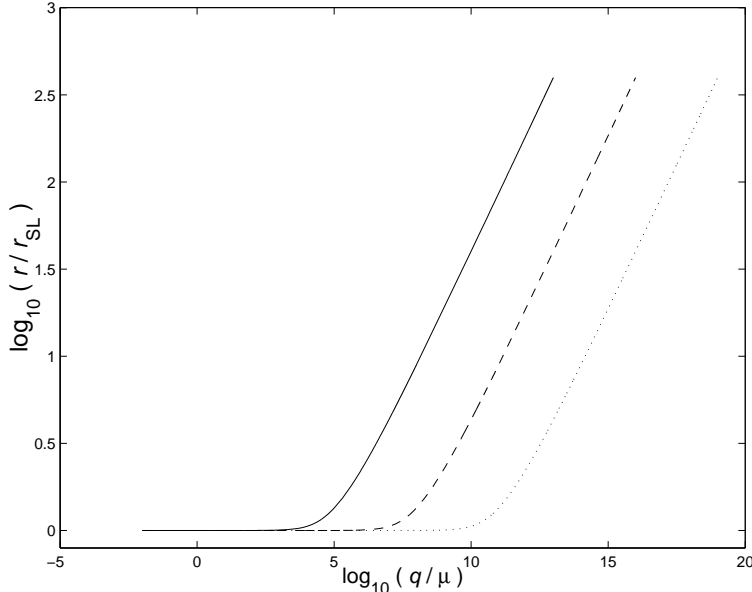


FIG. 9. The value of r/r_{SL} where a^{SF} equals a^{reg} as a function of q/μ on the equatorial plane of a Kerr black hole for $a/M = 0.5$, for three values of q/M : $q/M = 10^{-4}$ (solid line), $q/M = 10^{-7}$ (dashed line), and $q/M = 10^{-10}$ (dotted line).

- At large distances this SF agrees with the SF found by Gal'tsov [7], i.e., at large distances we find that

$$F_\varphi = \frac{q^2}{3} a \sin^2 \theta \frac{M^2 - Q^2}{r^4} \left\{ 1 + 3 \frac{M}{r} + \frac{1}{2} [3(5M^2 - Q^2) + 2a^2(1 - 3\cos^2 \theta)] \frac{1}{r^2} + O(r^{-3}) \right\}, \quad (83)$$

whose leading term agrees with Gal'tsov's result when $Q = 0$ (a Kerr black hole). Notice, that this expansion coincides with the asymptotic solution we found above in Eq. (51).

- The direction of the SF is in the direction of the spin of the black hole. Namely, in order to hold the particle static, the applied external force should be in the direction opposite to the spin. As noted by Gal'tsov [7], this direction can be explained as a tidal friction effect [29]: As the particle is accelerated in the direction of the black hole's spin, global conservation of angular momentum implies that the black hole is accelerated in the direction opposite to the spin, such that the black hole tends to spin down, and its rotational energy is being dissipated.
- When the particle's position is off the black hole's polar axis, the black hole is immersed in an external field, such that the entire configuration is not axially symmetric. From Hawking's theorem [26], stating that a stationary black hole must be either static or axisymmetric, it then follows that the black hole cannot remain stationary: it must evolve in time until it has become static or until it has achieved an axisymmetric orientation with the external field [27]. As the scalar field is stationary, there can't be any flux of energy down the event horizon, as measured by a static observer at infinity. Hence the black hole's mass M is unchanged. As the black hole's surface area \mathcal{A} , given by $\mathcal{A} = 8\pi M(M + \sqrt{M^2 - a^2})$ must increase, it follows that its angular momentum a must decrease, or that the black hole spins down. The dissipated rotational energy of the black hole does not escape to infinity, as the field is strictly static there. Instead, it flows down the hole as seen by a local, dragged observer. Specifically, a local observer who follows a timelike orbit must be dragged inside the ergosphere. Any such observer will see the field as time dependent, and will see a flux of energy down the event horizon, whose origin is in the black hole's rotational energy. However, when the particle is on the black hole's polar axis, the SF vanishes according to Eq. (80). In this case, the black hole is immersed in an axisymmetric field, such that it can remain stationary. The flux of angular momentum as viewed by a static distant observer vanishes, and the black hole's spin is unchanged.
- When Newton's constant G and the speed of light c are re-introduced, and the SF (80) is expanded in powers of G/c^2 , we find that

$$F_{\varphi}^{\text{SF}} = \frac{q^2}{3} \left(\frac{G}{c^2} \right)^2 a \sin^2 \theta \frac{M^2 - G^{-1} Q^2}{r^4} \left\{ \left[1 + 2 (1 - 3 \cos^2 \theta) \frac{a^2}{r^2} \right] + \left[3 + (5 - 14 \cos^2 \theta) \frac{a^2}{r^2} \right] \left(\frac{G}{c^2} \right) \frac{M}{r} + O \left(\frac{G}{c^2} \right)^2 \right\}, \quad (84)$$

such that we find that this SF is a (post)²-Newtonian effect. Note, that at the (post)²-Newtonian order the effect has contributions both from the r^{-4} and the r^{-6} terms. We also write explicitly in Eq. (84) the (post)³-Newtonian order of the SF.

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APPENDIX A:

EVALUATION OF $X_l(p, \theta)$ AND $\xi_l(p, \theta)$

When calculating the SF in the weak field region, we encounter the functions $X_l(p, \theta)$ and $\xi_l(p, \theta)$ defined as

$$X_l(p, \theta) = \sum_{m=-l}^l m^p Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) \quad (A1)$$

$$\xi_l(p, \theta) = \sum_{m=-l}^l m^p Y_{lm}^*(\theta, \varphi) \frac{\partial}{\partial \theta} Y_{lm}(\theta, \varphi). \quad (A2)$$

The spherical harmonics are of the form $Y_{lm}(\theta, \varphi) = \mathcal{Y}_{lm}(\theta) e^{im\varphi}$, where $\mathcal{Y}_{lm}(\theta)$ is a real-valued function. It follows that $X_l(p, \theta)$ and $\xi_l(p, \theta)$ are independent of φ . So we can take $\varphi = 0$ and write

$$X_l(p, \theta) = \sum_{m=-l}^l m^p [Y_{lm}(\theta, 0)]^2 \quad (A3)$$

$$\xi_l(p, \theta) = \sum_{m=-l}^l m^p Y_{lm}(\theta, 0) \frac{\partial}{\partial \theta} Y_{lm}(\theta, 0) = \frac{1}{2} \frac{\partial}{\partial \theta} X_l(p, \theta). \quad (A4)$$

The spherical harmonics have the property that $Y_{l,-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi)$, which implies that $[Y_{l,-m}(\theta, 0)]^2 = [Y_{lm}(\theta, 0)]^2$. So for $p \geq 1$, we have

$$X_l(p, \theta) = \sum_{m=1}^l m^p [1 + (-1)^p] [Y_{lm}(\theta, 0)]^2. \quad (A5)$$

It follows from Eqs. (A5) and (A4) that $X_l(p, \theta) = \xi_l(p, \theta) = 0$ if p is an odd integer.

To evaluate X_l , we use the addition formula for spherical harmonics:

$$\sum_{m=-l}^l Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) = \frac{2l+1}{4\pi} P_l(\nu), \quad (A6)$$

where P_l is the Legendre polynomial and

$$\nu = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'). \quad (A7)$$

It follows from Eqs. (A6), (A1) and $P_l(1) = 1$ that $X_l(0, \theta) = (2l+1)/(4\pi)$ and $\xi_l(0, \theta) = 0$. To compute X_l for $p > 1$, we set $\theta' = \theta$, $\varphi' = 0$ and differentiate Eq. (A6) p times with respect to φ . After some rearrangement, we obtain

$$X_l(p, \theta) = (-i)^p \frac{2l+1}{4\pi} \frac{\partial^p}{\partial \varphi^p} P_l(\nu) \Big|_{\varphi=0} \quad (\text{A8})$$

with $\nu = \cos^2 \theta + \sin^2 \theta \cos \varphi$. For simplicity, we only compute $X_l(p, \theta)$ for $p = 2$ and $p = 4$ here. Generalization to other values of p is straightforward. Eq. (A8) gives

$$X_l(2, \theta) = \frac{2l+1}{4\pi} \sin^2 \theta P'_l(1) \quad (\text{A9})$$

$$X_l(4, \theta) = \frac{2l+1}{4\pi} [\sin^2 \theta P'_l(1) + 3 \sin^4 \theta P''_l(1)] . \quad (\text{A10})$$

To compute $P'_l(1)$ and $P''_l(1)$, we use the recurrence relation

$$P'_{n+1}(x) = x P'_n(x) + (n+1) P_n(x) . \quad (\text{A11})$$

Differentiating the above equation, we have

$$P''_{n+1}(x) = x P''_n(x) + (n+2) P'_n(x) . \quad (\text{A12})$$

Evaluating the two equations at $x = 1$ and using the fact that $P_n(1) = 1$, we obtain

$$P'_{n+1}(1) - P'_n(1) = n+1 \quad (\text{A13})$$

$$P''_{n+1}(1) - P''_n(1) = (n+2) P'_l(1) . \quad (\text{A14})$$

Since $P_0(x) = 1$, we have $P'_0(1) = P''_0(1) = 0$. The difference equations (A13) and (A14) are then solved by summing over n on both sides from $n = 0$ to $(l-1)$. The results are

$$P'_l(1) = \frac{l(l+1)}{2} \quad (\text{A15})$$

$$P''_l(1) = \frac{(l-1)l(l+1)(l+2)}{8} . \quad (\text{A16})$$

Combining our results, we finally have

$$X_l(0, \theta) = \frac{2l+1}{4\pi} \quad (\text{A17})$$

$$X_l(2, \theta) = \frac{2l+1}{4\pi} \frac{l(l+1)}{2} \sin^2 \theta \quad (\text{A18})$$

$$X_l(4, \theta) = \frac{2l+1}{4\pi} \left[\frac{l(l+1)}{2} \sin^2 \theta + \frac{3}{8} (l-1)l(l+1)(l+2) \sin^4 \theta \right] . \quad (\text{A19})$$

The values of $\xi_l(p, \theta)$ can then be computed by Eq. (A4). The results are

$$\xi_l(0, \theta) = 0 \quad (\text{A20})$$

$$\xi_l(2, \theta) = \frac{2l+1}{4\pi} \frac{l(l+1)}{2} \sin \theta \cos \theta \quad (\text{A21})$$

$$\xi_l(4, \theta) = \frac{2l+1}{4\pi} \sin \theta \cos \theta \left[\frac{l(l+1)}{2} + \frac{3}{4} (l-1)l(l+1)(l+2) \sin^2 \theta \right] . \quad (\text{A22})$$

APPENDIX B:

EVALUATION OF f_φ^l

The φ -component of the SF f_φ^l is given by

$$f_\varphi^l = \sum_{m=-l}^l q \partial_\varphi \Phi^{lm} \quad (\text{B1})$$

$$= qS(r) \sqrt{M^2 - a^2 - Q^2} \sum_{m=-l}^l im \frac{\phi_1^{lm}(z) \phi_2^{lm}(z)}{W_z[\phi_1^{lm}, \phi_2^{lm}]} Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) . \quad (\text{B2})$$

All the quantities are evaluated at the position of the charge, which are assumed to be located at (r, θ, φ) . Here

$$W_z[\phi_1^{lm}, \phi_2^{lm}] = \phi_1^{lm} \frac{d\phi_2^{lm}}{dz} - \phi_2^{lm} \frac{d\phi_1^{lm}}{dz} \quad (\text{B3})$$

and the factor $\sqrt{M^2 - a^2 - Q^2}$ comes from the fact that

$$\frac{d}{dz} = \sqrt{M^2 - a^2 - Q^2} \frac{d}{dr} .$$

It is easy to show that $\Phi^{l,-m} = (\Phi^{lm})^*$, so f_φ^l is real. Hence we can write

$$\begin{aligned} f_\varphi^l &= qS(r) \sqrt{M^2 - a^2 - Q^2} \sum_{m=-l}^l \text{Re} \left\{ im \frac{\phi_1^{lm}(z) \phi_2^{lm}(z)}{W_z[\phi_1^{lm}, \phi_2^{lm}]} Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) \right\} \\ &= -qS(r) \sqrt{M^2 - a^2 - Q^2} \sum_{m=-l}^l \text{Im} \left\{ m \frac{\phi_1^{lm}(z) \phi_2^{lm}(z)}{W_z[\phi_1^{lm}, \phi_2^{lm}]} \right\} Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) . \end{aligned} \quad (\text{B4})$$

In Sect. II, we find that $\phi_1^{lm}(z) \propto P_l^{-\mu}(z)$ and $\phi_2^{lm}(z) \propto Q_l^\mu(z)$. However, the function $Q_l^{-\mu}(z)$ is related to $Q_l^\mu(z)$ by [25]

$$Q_l^{-\mu}(z) = e^{-2i\mu\pi} \frac{\Gamma(l - \mu + 1)}{\Gamma(l + \mu + 1)} Q_l^\mu(z) . \quad (\text{B5})$$

Hence $Q_l^{-\mu}$ and Q_l^μ are linearly dependent, and so we can as well choose $\phi_2^{lm} \propto Q_l^{-\mu}$. In this Appendix, we set

$$\phi_1^{lm}(z) = P_l^{-\mu}(z) \quad (\text{B6})$$

$$\phi_2^{lm}(z) = Q_l^{-\mu}(z) = \frac{e^{-i\mu\pi}}{(2l+1)!!} \frac{\Gamma(l - \mu + 1)}{z^{l+1}} K_l^{-\mu}(z) , \quad (\text{B7})$$

where $(2l+1)!! = 1 \cdot 3 \cdot 5 \cdots (2l+1)$ and

$$K_l^\mu(z) = \left(1 - \frac{1}{z^2}\right)^{\mu/2} {}_2F_1\left(1 + \frac{l+\mu}{2}, \frac{l+\mu+1}{2}; l + \frac{3}{2}; \frac{1}{z^2}\right) . \quad (\text{B8})$$

We have used Eq. (21) and the fact that

$$\Gamma\left(l + \frac{3}{2}\right) = \frac{(2l+1)!!}{2^{l+1}} \sqrt{\pi} \quad (\text{B9})$$

to obtain the second equality in Eq. (B7). Since $\mu = im\gamma$ is purely imaginary, we have $K_l^{-\mu} = (K_l^\mu)^*$. It follows from Eqs. (B5) and (B7) that

$$K_l^{-\mu}(z) = [K_l^\mu(z)]^* = K_l^\mu(z) , \quad (\text{B10})$$

Hence K_l^μ is a real-valued function, although it is not obvious to see this from Eq. (B8). $Q_l^\mu(z)$ is then just a complex factor independent of z times a real-valued function, so we can choose $\phi_2^{lm}(z)$ to be real. In fact, the $\phi_2^{lm}(z)$ in Eq. (34) is real, because it is equal to $K_l^\mu(z)/z^{l+1}$. In this Appendix, however, we stick to the choice in Eq. (B7).

The Wronskian is given by [25]

$$W_z[\phi_1^{lm}, \phi_2^{lm}] = \frac{e^{-i\mu\pi} 2^{-2\mu}}{1 - z^2} \frac{\Gamma\left(\frac{l-\mu}{2} + 1\right) \Gamma\left(\frac{l-\mu+1}{2}\right)}{\Gamma\left(\frac{l+\mu}{2} + 1\right) \Gamma\left(\frac{l+\mu+1}{2}\right)}$$

$$= \frac{e^{-i\mu\pi}}{1-z^2} \frac{\Gamma(l-\mu+1)}{\Gamma(l+\mu+1)}, \quad (\text{B11})$$

where we have used the identity [25]

$$\Gamma(2x) = \frac{2^{2x-\frac{1}{2}}}{\sqrt{2\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right). \quad (\text{B12})$$

Combining Eqs. (B6), (B7), (B10) and (B11), we obtain

$$\frac{\phi_1^{lm}(z)\phi_2^{lm}(z)}{W_z[\phi_1^{lm}, \phi_2^{lm}]} = \left[\frac{1}{z^{l+1}} \frac{1-z^2}{(2l+1)!!} K_l^\mu(z) \right] [\Gamma(l+\mu+1)P_l^{-\mu}(z)]. \quad (\text{B13})$$

Note that the quantity in the first bracket is real, while that in the second is complex. To extract the imaginary part, we subtract from Eq. (B13) its complex conjugate and then divide by $2i$:

$$\begin{aligned} \text{Im} \left[\frac{\phi_1^{lm}(z)\phi_2^{lm}(z)}{W_z[\phi_1^{lm}, \phi_2^{lm}]} \right] &= \left[\frac{1}{z^{l+1}} \frac{1-z^2}{(2l+1)!!} K_l^\mu(z) \right] \left[\frac{\Gamma(l+\mu+1)P_l^{-\mu}(z) - \Gamma(l-\mu+1)P_l^\mu(z)}{2i} \right] \\ &= \left[\frac{1}{z^{l+1}} \frac{1-z^2}{(2l+1)!!} K_l^\mu(z) \right] \left[-\frac{\Gamma(l+\mu+1)\Gamma(l-\mu+1)}{\pi(2l+1)!!} \frac{\sinh(\pi m\gamma)}{z^{l+1}} K_l^\mu(z) \right], \end{aligned} \quad (\text{B14})$$

where we have used the fact that $\mu = im\gamma$ and [25]

$$P_l^\mu(z) = \frac{\Gamma(l+\mu+1)}{\Gamma(l-\mu+1)} \left[P_l^{-\mu}(z) + \frac{2}{\pi} e^{i\pi\mu} \sin(\mu\pi) Q_l^{-\mu}(z) \right]. \quad (\text{B15})$$

The product $\Gamma(l+\mu+1)\Gamma(l-\mu+1)$ can be evaluated by the identities $\Gamma(x+1) = x\Gamma(x)$ and [25]

$$\Gamma(iy)\Gamma(-iy) = \frac{\pi}{y \sinh(\pi y)} \quad (\text{B16})$$

for real y . The result is

$$\text{Im} \left[\frac{\phi_1^{lm}(z)\phi_2^{lm}(z)}{W_z[\phi_1^{lm}, \phi_2^{lm}]} \right] = \frac{m\gamma}{z^{2l}} \left(1 - \frac{1}{z^2} \right) \left[\prod_{j=1}^l (m^2\gamma^2 + j^2) \right] \left[\frac{K_l^\mu(z)}{(2l+1)!!} \right]^2. \quad (\text{B17})$$

Substituting the above formula into Eq. (B4), we find

$$f_\varphi^l = -\frac{qS(r)}{z^{2l}} \gamma \sqrt{M^2 - a^2 - Q^2} \left(1 - \frac{1}{z^2} \right) \sum_{m=-l}^l m^2 \left[\prod_{j=1}^l (m^2\gamma^2 + j^2) \right] \left[\frac{K_l^\mu(z)}{(2l+1)!!} \right]^2 Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi), \quad (\text{B18})$$

and substituting for the value of γ we finally obtain

$$f_\varphi^l = -\frac{qS(r)}{z^{2l}} a \left(1 - \frac{1}{z^2} \right) \sum_{m=-l}^l m^2 \left[\prod_{j=1}^l (m^2\gamma^2 + j^2) \right] \left[\frac{K_l^\mu(z)}{(2l+1)!!} \right]^2 Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi). \quad (\text{B19})$$

APPENDIX C:

NUMERICAL EVALUATION OF THE SELF FORCE

A very effective way to evaluate the associated Legendre functions $P_l^{-\mu}(z)$ and $Q_l^\mu(z)$ numerically is by their representations using hypergeometric functions. Specifically, we rewrite the associated Legendre functions as

$$P_l^{-\mu}(z) = \frac{1}{\Gamma(1+\mu)} \left(\frac{z+1}{z-1} \right)^{-\mu/2} {}_2F_1 \left(-l, l+1; 1+\mu; \frac{1-z}{2} \right) \quad (\text{C1})$$

and

$$Q_l^\mu(z) = e^{i\mu\pi} \frac{\sqrt{\pi}}{2^{l+1}} \frac{\Gamma(1+l+\mu)}{\Gamma(l+3/2)} \frac{(z^2-1)^{\mu/2}}{z^{1+l+\mu}} {}_2F_1\left(1 + \frac{l+\mu}{2}, \frac{1+l+\mu}{2}; l + \frac{3}{2}; \frac{1}{z^2}\right). \quad (\text{C2})$$

In all the expressions we need to evaluate, we only have products of $P_l^{-\mu}(z)$ and $Q_l^\mu(z)$ divided by their Wronskian determinant. Hence, we do not need to evaluate the constant factors in Eqs. (C1) and (C2). The derivatives of $P_l^{-\mu}(z)$ and $Q_l^\mu(z)$, which are needed both for the Wronskian determinant and for the gradient of the field in the SF computation, can be computed using the relation

$$\frac{d}{dz} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z). \quad (\text{C3})$$

It is convenient to evaluate the hypergeometric functions in Eqs. (C1-C3) using their Gauss series representation, i.e.,

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n \quad |z| < 1, \quad (\text{C4})$$

where $(a)_n$ is Pochhammer's symbol. This is a useful numerical approach because of the following. For the evaluation of $P_l^{-\mu}(z)$ we need to compute a hypergeometric function for which the Gauss series is reduced to a polynomial of degree l in $(1-z)/2$, thanks to the first variable of the hypergeometric function being a negative integer. Hence, we only need to sum over a finite number of terms, and there is no truncation error involved. We are also unrestricted by the radius of convergence of (C4). This fortunate circumstance does not change for the calculation of the derivative. For the computation of $Q_l^\mu(z)$ we benefit from the argument being positive and smaller than unity. Consequently, in the Gauss series we have a sum over terms which are a (combinatorial factor) $\times z^{-2n}$, where $z > 1$ and n is a positive integer. This guarantees that the series converges fast, such that in practice we do not need to sum over too many terms to ensure a given accuracy. In practice we evaluate all hypergeometric function to accuracy of 1 part in 10^{16} . Our computation does not become more complicated because of μ being imaginary.

An alternative approach for computing $Q_l^\mu(z)$ is to use its integral representation, given by

$$Q_l^\mu(z) = e^{i\mu\pi} \frac{\Gamma(l+\mu+1)}{2^{l+1}\Gamma(l+1)} (z^2-1)^{\mu/2} \int_{-1}^1 dt \frac{(1-t^2)^l}{(z-t)^{l+\mu+1}}, \quad (\text{C5})$$

which is efficient numerically, because the integration interval is compact, and the integrand has pathologies neither inside the interval nor at its boundaries.

The spherical harmonics can be computed either from the associated Legendre functions computed using the hypergeometric representation above (which is not very convenient because of the large arguments of the gamma functions, which are needed to be computed now), or, alternatively, from the recurrence relations for the associated Legendre functions as given in [32]. This latter approach is stable and accurate also for the very large values of l, m we need (≈ 1000).

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